Solutions

1. Given, $R = \{(a, a^3): a \text{ is a prime number less than } 5\}$

We know that, 2 and 3 are the prime numbers less than 5. So, a can take values 2 and 3.

Thus, $R = \{(2, 2^3), (3, 3^3)\} = \{(2, 8), (3, 27)\}$

Hence, the range of R is $\{8, 27\}$. **(1)**

2. Given, functions $f: \{1, 3, 4\} \to \{1, 2, 5\}$ and $g: \{1, 2, 5\} \to \{1, 3\}$ are defined as

$f = \{(1, 2), (3, 5), (4, 1)\}$ and $g = \{(1, 3), (2, 3), (5, 1)\}$

Therefore, $f(1) = 2$, $f(3) = 5$, $f(4) = 1$

and $g(1) = 3$, $g(2) = 3$, $g(5) = 1$

Now, $gof: \{1, 3, 4\} \to \{1, 3\}$ and it is defined as

$gof(1) = g[(f(1)] = g(2) = 3$

$gof(3) = g[f(3)] = g(5) = 1$

$gof(4) = g[f(4)] = g(1) = 3$

$\therefore \quad gof = \{(1, 3), (3, 1), (4, 3)\}$ **(1)**

3. Given, $R = \{(a, b): 2 \text{ divides } (a - b)\}$

and $A = \{0, 1, 2, 3, 4, 5\}$

Clearly, $[0] = \{b \in A: (0, b) \in R\}$

$= \{b \in A: 2 \text{ divides } (0 - b)\}$

$= \{b \in A: 2 \text{ divides } (-b)\} = \{0, 2, 4\}$

Hence, equivalence class of $[0] = \{0, 2, 4\}$. **(1)**

4. Given, the relation R is defined on the set of natural numbers, i.e. N as

$$R = \{(x, y): x + 2y = 8\}$$

To find the range of R, $x + 2y = 8$ can be rewritten as $y = \dfrac{8 - x}{2}$.

On putting $x = 2$, we get $y = \dfrac{8 - 2}{2} = 3$

On putting $x = 4$, we get $y = \dfrac{8 - 4}{2} = 2$

On putting $x = 6$, we get $y = \dfrac{8 - 6}{2} = 1$

As, $x, y \in N$, therefore $R = \{(2, 3), (4, 2), (6, 1)\}$.

Hence, the range of relation R is $\{3, 2, 1\}$. **(1)**

NOTE For $x = 1, 3, 5, 7, 9, \ldots$, we do not get y as natural number.

5. Given, $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$ and $f : A \to B$ is defined as $f = \{(1, 4), (2, 5), (3, 6)\}$ i.e. $f(1) = 4$, $f(2) = 5$ and $f(3) = 6$.

It can be seen that the images of distinct elements of A under f are distinct. So, f is one-one. **(1)**

6. Given, $f(x) = 3x + 2$

$\therefore \quad f[f(x)] = f(3x + 2) = 3(3x + 2) + 2$

$= 9x + 6 + 2 = 9x + 8$ **(1)**

7. Given, $f(x) = |x|$, $g(x) = |5x - 2|$

$\therefore \quad fog\,(x) = f[g(x)] = f\{|5x - 2|\}$

$= ||5x - 2|| = |5x - 2| \quad [\because \; ||x|| = |x|]$ **(1)**

8. Given, $f(x) = 8x^3$ and $g(x) = x^{1/3}$

$\therefore \; fog\,(x) = f[g(x)] = f(x^{1/3}) = 8(x^{1/3})^3 = 8x$ **(1)**

9. We know that for a relation to be transitive, $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$.

Here, $(1, 2) \in R$ and $(2, 1) \in R$ but $(1, 1) \notin R$.

$\therefore \quad R$ is not transitive. **(1)**

10. Firstly, redefine the function by using the definition of modulus function, i.e. by using $|x| = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}$ Further, simplify it to get the range.

Given, function is $f(x) = \dfrac{|x-1|}{x-1}$, $x \ne 1$.

The above function can be written as

$$f(x) = \begin{cases} \dfrac{x-1}{x-1}, & \text{if } x > 1 \\ -\dfrac{(x-1)}{x-1}, & \text{if } x < 1 \end{cases}$$

$$\Rightarrow \quad f(x) = \begin{cases} 1, & \text{if } x > 1 \\ -1, & \text{if } x < 1 \end{cases}$$

Hence, the range of $f(x)$ is $\{-1, 1\}$. **(1)**

11. Given function is $f : R \to R$ such that $f(x) = (3 - x^3)^{1/3}$.

Now, $fof\,(x) = f[f(x)] = f[(3 - x^3)^{1/3}]$

$= [3 - \{(3 - x^3)^{1/3}\}^3]^{1/3}$

$= [3 - (3 - x^3)]^{1/3} = (x^3)^{1/3} = x$ **(1)**

12. Given, $f(x) = \dfrac{3x - 4}{5}$ is an invertible function.

Let $\quad y = \dfrac{3x - 4}{5} \Rightarrow 5y = 3x - 4$

$\Rightarrow \quad 3x = 5y + 4 \Rightarrow x = \dfrac{5y + 4}{3}$

$\therefore \quad f^{-1}(y) = \dfrac{5y + 4}{3}$ or $f^{-1}(x) = \dfrac{5x + 4}{3}$ **(1)**

13. Given, $f(x) = \sin x$ and $g(x) = 5x^2$.

$\therefore \quad gof(x) = g[f(x)] = g(\sin x)$

$= 5(\sin x)^2 = 5\sin^2 x$ **(1)**

14. Given, $f(x) = 27x^3$ and $g(x) = x^{1/3}$.

Now, $gof(x) = g[f(x)] = g(27x^3)$

$= (27x^3)^{1/3} = (27)^{1/3} \cdot (x^3)^{1/3}$

$= (3^3)^{1/3} \cdot (x^3)^{1/3} = 3x$

$\therefore \quad gof(x) = 3x$ **(1)**

15. Do same as Q.No. 12. $\left[\textbf{Ans. } \dfrac{x+4}{3}\right]$

16. The relation R on set $A = \{1, 2, 3, 4, 5, 6\}$ is defined as $(a, b) \in R$ iff $b = a + 1$.

Therefore, $R = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)\}$ **(1)**

Clearly, $(a, a) \notin R$ for any $a \in A$. So, R is not reflexive on A.

We observe that $(1, 2) \in R$ but $(2, 1) \notin R$.

So, R is not symmetric. **(1½)**

We also observe that $(1, 2) \in R$ and $(2, 3) \in R$ but $(1, 3) \notin R$. So, R is not transitive. **(1½)**

17. Given, $f : N \to Y$ defined as $f(x) = 4x + 3$, where $Y = \{y \in N : y = 4x + 3, x \in N\}$. Consider an arbitrary element $y \in Y$. Then, $y = 4x + 3$, for some $x \in N$

$\Rightarrow \quad y - 3 = 4x \Rightarrow x = \dfrac{y - 3}{4}$ **(1)**

Suppose, a function $g : Y \to N$, given by

$$g(y) = \frac{y - 3}{4}$$

Now, $gof(x) = g\{f(x)\} = g(4x + 3)$

$= \dfrac{4x + 3 - 3}{4} = x$ **(1)**

and $\quad fog\,(y) = f\{g(y)\} = f\left(\dfrac{y - 3}{4}\right)$

$= 4\left(\dfrac{y - 3}{4}\right) + 3 = y$ **(1)**

Here, $gof\,(x) = x, \forall x \in N$; therefore $gof = I_N$ and $fog(y) = y, \forall\, y \in Y$; therefore $fog = I_Y$

So, f is invertible and $f^{-1} = g$,

i.e. $\quad f^{-1}(y) = \dfrac{y - 3}{4}$

or $\quad f^{-1}(x) = \dfrac{x - 3}{4}$ **(1)**

18. Given a relation $R = \{(a, b) : a \le b\}$ on IR (the set of real numbers).

Reflexivity

Since, $a \le a$ is true for all value of $a \in IR$.

$\therefore \quad (a, a) \in R \forall a \in IR$

Hence, the given relation is reflexive. **(1)**

Transitivity

Let $(a, b) \in R$ and $(b, c) \in R$ be any arbitrary elements.

Then, we have $a \le b$ and $b \le c$

$\Rightarrow \quad a \le b \le c$

$\Rightarrow \quad a \le c$

$\Rightarrow \quad (a, c) \in R$

Hence, the given relation is transitive. **(1½)**

Symmetricity

Note that $(2, 3) \in R$ as $2 < 3$

but $(3, 2) \notin R$ as $3 \not< 2$

Hence, the given relation is not symmetric.

Hence proved. (1½)

19. Let $x, y \in N$ such that

$$f(x) = f(y)$$

$\Rightarrow \quad x^2 + x + 1 = y^2 + y + 1$

$\Rightarrow \quad (x - y)(x + y + 1) = 0 \qquad [\because x + y + 1 \ne 0]$

$\Rightarrow \quad x = y$

$\therefore \quad f : N \to N$ is one-one **(1)**

f is not onto because $x^2 + x + 1 \ge 3, \forall x \in N$

and so, 1,2 does not have their pre images. **(1)**

Now, if S is the range of f, then $f : N \to S$ is one-one, onto and hence invertible.

$\Rightarrow \quad fof^{-1}(x) = x, \forall x \in S$

$\Rightarrow \quad f(f^{-1}(x)) = x, \forall x \in S$

$\Rightarrow \quad (f^{-1}(x))^2 + (f^{-1}(x)) + 1 = x, \forall x \in S$

$\Rightarrow \quad (f^{-1}(x))^2 + f^{-1}(x) + 1 - x = 0,$

which is quadratic in $f^{-1}(x)$ **(1)**

$$\Rightarrow \quad f^{-1}(x) = \frac{-1 \pm \sqrt{1 - 4(1 - x)}}{2}$$

$$\Rightarrow \quad = \frac{-1 \pm \sqrt{4x - 3}}{2}$$

But $f^{-1}(x) \in N$

$$\therefore \quad f^{-1}(x) = \frac{-1 + \sqrt{4x - 3}}{2} \quad \textbf{(1)}$$

20. Given, $f : W \to W$ is defined as

$$f(x) = \begin{cases} x - 1, & \text{if } x \text{ is odd} \\ x + 1, & \text{if } x \text{ is even} \end{cases}$$

One-one function Let $x_1, x_2 \in W$ be any two numbers such that $f(x_1) = f(x_2)$.

Case I When x_1 and x_2 are odd.

Then, $\quad f(x_1) = f(x_2) \Rightarrow x_1 - 1 = x_2 - 1$

$\Rightarrow \quad x_1 = x_2$

Case II When x_1 and x_2 are even.

Then, $\quad f(x_1) = f(x_2)$

$\Rightarrow \quad x_1 + 1 = x_2 + 1$

$\Rightarrow \quad x_1 = x_2$

Thus, in both cases,

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \quad \textbf{(1)}$$

Case III When x_1 is odd and x_2 is even.

Then, $x_1 \ne x_2$.

Also, $f(x_1)$ is even and $f(x_2)$ is odd.

So, $f(x_1) \ne f(x_2)$

Thus, $x_1 \ne x_2 \Rightarrow f(x_1) \ne f(x_2)$

Case IV When x_1 is even and x_2 is odd.

Then, $x_1 \ne x_2$.

Also, $f(x_1)$ is odd and $f(x_2)$ is even.

So, $f(x_1) \ne f(x_2)$

Thus, $x_1 \ne x_2 \Rightarrow f(x_1) \ne f(x_2)$

Hence, from cases I, II, III and IV we can observe that, $f(x)$ is a one-one function. **(1)**

Onto function Clearly, any odd number $2y+1$ in the codomain W, is the image of $2y$ in the domain W.

Also, any even number $2y$ in the codomain W, is the image of $2y + 1$ in the domain W.

Thus, every element in W (codomain) has a pre-image in W (domain).

So, f is onto.

Therefore, f is bijective and so it is invertible. **(1)**

Let $\quad f(x) = y$

$\Rightarrow \quad x - 1 = y$, if x is odd

and $\quad x + 1 = y$, if x is even

$$\therefore \quad x = \begin{cases} y + 1, & \text{if } y \text{ is even} \\ y - 1, & \text{if } y \text{ is odd} \end{cases}$$

$$\text{or} \quad f^{-1}(x) = \begin{cases} x + 1, & \text{if } x \text{ is even} \\ x - 1, & \text{if } x \text{ is odd} \end{cases} \quad \textbf{(1)}$$

21. Given, $f(x) = |x| + x$ and $g(x) = |x| - x$ for all $x \in R$.

$$\Rightarrow \quad f(x) = \begin{cases} x + x, & x \geq 0 \\ -x + x, & x < 0 \end{cases}$$

and $$g(x) = \begin{cases} x - x, & x \geq 0 \\ -x - x, & x < 0 \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} 2x, & x \geq 0 \\ 0, & x < 0 \end{cases} \text{ and } g(x) = \begin{cases} 0, & x \geq 0 \\ -2x, & x < 0 \end{cases}$$ **(1)**

Thus, for $x \geq 0$, $gof(x) = g(f(x)) = g(2x) = 0$

and for $x < 0$, $gof(x) = g(f(x)) = g(0) = 0$

$$\Rightarrow \quad gof(x) = 0, \forall\, x \in R$$ **(1½)**

Similarly, for $x \geq 0$, $fog(x) = f(g(x)) = f(0) = 0$

and for $x < 0$, $fog(x) = f(g(x)) = f(-2x)$

$$= 2(-2x) = -4x$$

$$\Rightarrow \quad fog(x) = \begin{cases} 0, & x \geq 0 \\ -4x, & x < 0 \end{cases}$$ **(1½)**

22. Given, $R = \{(x, y) : x \in N, y \in N \text{ and } 2x + y = 24\}$

$\therefore \quad y = 24 - 2x$

Now, $x = 1 \Rightarrow y = 22$;

$x = 2 \Rightarrow y = 20$;

$x = 3 \Rightarrow y = 18$; $x = 4 \Rightarrow y = 16$;

$x = 5 \Rightarrow y = 14$; $x = 6 \Rightarrow y = 12$;

$x = 7 \Rightarrow y = 10$; $x = 8 \Rightarrow y = 8$

$x = 9 \Rightarrow y = 6$; $x = 10 \Rightarrow y = 4$

and $x = 11 \Rightarrow y = 2$ **(1)**

So, domain of $R = \{1, 2, 3, \ldots, 11\}$

and range of $R = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22\}$

and $R = \{(1, 22), (2, 20), (3, 18), (4, 16), (5, 14), (6, 12), (7, 10), (8, 8), (9, 6), (10, 4), (11, 2)\}$ **(1)**

Reflexive Since, for $1 \in$ domain of R, $(1, 1) \notin R$.

So, R is not reflexive.

Symmetric We observe that $(1, 22) \in R$ but $(22, 1) \notin R$. So, R is not symmetric.

Transitive We observe that $(7, 10) \in R$ and $(10, 4) \in R$ but $(7, 4) \notin R$. So, R is not transitive. **(1)**

Thus, R is neither reflexive nor symmetric nor transitive.

So, R is not an equivalence relation. **(1)**

23. Given, a function $f : A \to B$, where $A = R - \{3\}$ and $B = R - \{1\}$, defined by $f(x) = \dfrac{x - 2}{x - 3}$.

One-one function

Let $x_1, x_2 \in A$ such that $f(x_1) = f(x_2)$

Then, $$\frac{x_1 - 2}{x_1 - 3} = \frac{x_2 - 2}{x_2 - 3}$$

$$\Rightarrow \quad (x_1 - 2)(x_2 - 3) = (x_2 - 2)(x_1 - 3)$$

$$\Rightarrow \quad x_1 x_2 - 3x_1 - 2x_2 + 6 = x_1 x_2 - 3x_2 - 2x_1 + 6$$

$$\Rightarrow \quad -3x_1 - 2x_2 = -3x_2 - 2x_1$$

$$\Rightarrow \quad -3(x_1 - x_2) + 2(x_1 - x_2) = 0$$

$$\Rightarrow \quad -(x_1 - x_2) = 0$$

$$\Rightarrow \quad x_1 - x_2 = 0 \Rightarrow x_1 = x_2$$

Thus, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2, \forall\, x_1, x_2 \in A$

So, $f(x)$ is a one-one function. **(1½)**

Onto function Let $y \in B = R - \{1\}$ be any arbitrary element.

Then, $f(x) = y$

$$\Rightarrow \quad \frac{x - 2}{x - 3} = y \Rightarrow x - 2 = xy - 3y$$

$$\Rightarrow \quad x - xy = 2 - 3y$$

$$\Rightarrow \quad x(1 - y) = 2 - 3y$$

$$\Rightarrow \quad x = \frac{2 - 3y}{1 - y} \text{ or } x = \frac{3y - 2}{y - 1} \qquad \text{...(i)}$$

Clearly, $x = \dfrac{3y - 2}{y - 1}$ is a real number for all $y \neq 1$.

Also, $\dfrac{3y - 2}{y - 1} \neq 3$ $\left[\because \dfrac{3y - 2}{y - 1} = 3 \Rightarrow 3y - 2 = 3y - 3 \Rightarrow 2 = 3 \text{ which is absurd.}\right]$

Thus, for each $y \in B$, there exists $x = \dfrac{3y - 2}{y - 1} \in A$

such that $$f(x) = f\left(\frac{3y - 2}{y - 1}\right) = \frac{\left(\frac{3y - 2}{y - 1}\right) - 2}{\frac{3y - 2}{y - 1} - 3}$$

$$= \frac{3y - 2 - 2y + 2}{3y - 2 - 3y + 3} = y$$

Hence, $f(x)$ is an onto function. **(1½)**

Therefore, $f(x)$ is a bijective function.

From Eq. (i), we get

$$f^{-1}(y) = \frac{3y - 2}{y - 1} \text{ or } f^{-1}(x) = \frac{3x - 2}{x - 1}$$

which is the inverse function of $f(x)$. **(1)**

24. Given a relation R in $A \times A$, where $A = \{1, 2, 3, \ldots, 9\}$, defined as $(a, b)\, R\, (c, d)$, if $a + d = b + c$.

Reflexive Let (a, b) be any arbitrary element of $A \times A$ i.e. $(a, b) \in A \times A$, where $a, b \in A$.

Now, as $a+b=b+a$

[$\because$ addition is commutative]

$\therefore$ $(a, b)R(a, b)$

So, R is reflexive. **(1/2)**

Symmetric Let $(a, b), (c, d) \in A \times A$, such that $(a, b)R(c, d)$. Then, $a+d=b+c$

$\Rightarrow$ $b+c=a+d \Rightarrow c+b=d+a$

[$\because$ addition is commutative]

$\Rightarrow (c, d) R(a, b)$

So, R is symmetric. **(1/2)**

Transitive Let $(a, b), (c, d), (e, f) \in A \times A$ such that $(a, b) R(c, d)$ and $(c, d) R(e, f)$.

Then, $a+d=b+c$ and $c+f=d+e$

On adding the above equations, we get

$a+d+c+f=b+c+d+e$

$\Rightarrow$ $a+f=b+e \Rightarrow (a, b)R(e, f)$

So, R is transitive. **(1/2)**

Thus, R is reflexive, symmetric and transitive. Hence, R is an equivalence relation. **(1)**

Now, for $[(2, 5)]$, we will find $(c, d) \in A \times A$ such that $2+d=5+c$ or $d-c=3$ **(1/2)**

Clearly, $(2, 5)R(1, 4)$ as $4-1=3$

$(2, 5)R(2, 5)$ as $5-2=3$

$(2, 5)R(3, 6)$ as $6-3=3$

$(2, 5)R(4, 7)$ as $7-4=3$

$(2, 5)R(5, 8)$ as $8-5=3$

and $(2, 5)R(6, 9)$ as $9-6=3$

Hence, equivalence class $[(2, 5)]$

$=\{(1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)\}$. **(1)**

25. Given, $f: R \rightarrow R$ and $g: R \rightarrow R$ defined as

$$f(x)=x^2+2 \text{ and } g(x)=\frac{x}{x-1}; x \neq 1$$

Since, range $f \subseteq$ domain g and range $g \subseteq$ domain f

$\therefore$ fog and gof exist.

For any $x \in R-\{1\}$, we have $(fog)(x)=f[g(x)]$

$$=f\left[\frac{x}{x-1}\right]=\left(\frac{x}{x-1}\right)^2+2$$

$$=\frac{x^2+2(x-1)^2}{(x-1)^2}=\frac{x^2+2(x^2+1-2x)}{(x-1)^2}=\frac{3x^2+2-4x}{(x-1)^2}$$

$\therefore$ $fog: R \rightarrow R$ is defined by

$$(fog)(x)=\frac{3x^2-4x+2}{(x-1)^2}, x \neq 1 \quad \text{...(i) (1)}$$

For any $x \in R$, we have

$(gof)(x)=g[f(x)]$

$$=g(x^2+2)=\frac{x^2+2}{(x^2+2)-1}=\frac{x^2+2}{x^2+1}$$

$\therefore$ $gof: R \rightarrow R$ is defined by

$$(gof)(x)=\frac{x^2+2}{x^2+1} \quad \text{...(ii) (1)}$$

On putting $x=2$ in Eq. (i), we get

$$fog(2)=\frac{3 \times (2)^2-4(2)+2}{(2-1)^2}=\frac{3 \times 4-8+2}{(1)^2}$$

$$=12-8+2=6 \quad \text{(1)}$$

On putting $x=-3$ in Eq. (ii), we get

$$(gof)(-3)=\frac{(-3)^2+2}{(-3)^2+1}=\frac{9+2}{9+1}=\frac{11}{10} \quad \text{(1)}$$

26. Do same as Q. No. 23. $\left[\textbf{Ans. } \frac{2x-1}{x-1}\right]$

27. Given, $f(x)=\dfrac{4x+3}{6x-4}$

where, $x \in A=R-\left\{\frac{2}{3}\right\}$

One-one function Let $x_1, x_2 \in A=R-\left\{\frac{2}{3}\right\}$ such that $f(x_1)=f(x_2)$.

Then, $\dfrac{4x_1+3}{6x_1-4}=\dfrac{4x_2+3}{6x_2-4}$

$\Rightarrow$ $(4x_1+3)(6x_2-4)=(4x_2+3)(6x_1-4)$

$\Rightarrow$

$24x_1x_2-16x_1+18x_2-12=24x_1x_2-16x_2+18x_1-12$

$\Rightarrow$ $-34x_1=-34x_2$

$\Rightarrow$ $x_1=x_2$

So, f is one-one function. **(1)**

Onto function Let y be an arbitrary element of A (codomain).

Then, $f(x)=y$

$\Rightarrow$ $\dfrac{4x+3}{6x-4}=y$

$\Rightarrow$ $4x+3=6xy-4y$

$\Rightarrow$ $4x-6xy=-4y-3$

$\Rightarrow$ $x(4-6y)=-(4y+3)$

$\Rightarrow$ $x=\dfrac{-(4y+3)}{4-6y}$

$\Rightarrow$ $x=\dfrac{4y+3}{6y-4}$

Clearly, $x=\dfrac{4y+3}{6y-4}$ is a real number for all $y \neq \dfrac{4}{6}=\dfrac{2}{3}$. **(1)**

Also, $\frac{4y+3}{6y-4} \neq \frac{2}{3}$ $\left[\because \frac{4y+3}{6y-4} = \frac{2}{3} \Rightarrow 12y+9 = 12y-8 \Rightarrow 9 = -8, \text{ which is absurd}\right]$

Thus, for each $y \in A$ (codomain), there exists $x = \frac{4y+3}{6y-4} \in A$ (domain) such that

$$f(x) = f\left(\frac{4y+3}{6y-4}\right) = \frac{4\left(\frac{4y+3}{6y-4}\right)+3}{6\left(\frac{4y+3}{6y-4}\right)-4}$$

$$= \frac{16y+12+18y-12}{24y+18-24y+16} = \frac{34y}{34} = y$$

Hence, f is onto function. **(1)**

Since, f is bijective function, so its inverse exists.

$\therefore \quad x = f^{-1}(y) = \frac{3+4y}{6y-4}$

or $f^{-1}(x) = \frac{3+4x}{6x-4}, x \neq \frac{2}{3}$ **(1)**

28. To show $f(x)$ is an invertible function, we will show that f is both one-one and onto function.

Here, function $f: R^+ \to [4, \infty)$ is given by $f(x) = x^2 + 4$.

One-one function Let $x, y \in R^+$, such that $f(x) = f(y)$.

$\Rightarrow \quad x^2 + 4 = y^2 + 4 \quad \Rightarrow x^2 = y^2 \Rightarrow \quad x = y$

[$\because$ we take only positive sign as $x, y \in R^+$]

Therefore, f is a one-one function. **(1½)**

Onto function For $y \in [4, \infty)$,

then there exists $x \in R^+$ such that $f(x) = y$

$\Rightarrow \quad y = x^2 + 4$

$\Rightarrow \quad x^2 = y - 4 \geq 0 \quad [\because y \geq 4]$

$\Rightarrow \quad x = \sqrt{y-4} \geq 0$

[we take only positive sign, as $x \in R^+$]

Therefore, for any $y \in R^+$ (codomain), there exists $x = \sqrt{y-4} \in R^+$ (domain) such that

$f(x) = f(\sqrt{y-4}) = (\sqrt{y-4})^2 + 4 = y - 4 + 4 = y$

Therefore, f is onto function.

Since, f is one-one and onto and therefore f^{-1} exists. **(1½)**

$\therefore \quad x = f^{-1}(y) = \sqrt{y-4}$

or $\quad f^{-1}(x) = \sqrt{x-4}, x \in [4, \infty)$ **(1)**

Alternate Method

Let us define $g: [4, \infty) \to R^+$,

by $g(y) = \sqrt{y-4}$.

Now, $gof(x) = g(f(x)) = g(x^2 + 4)$

$= \sqrt{(x^2+4)-4} = \sqrt{x^2} = x$ **(1½)**

and $\quad fog(y) = f[g(y)] = f(\sqrt{y-4})$

$= (\sqrt{y-4})^2 + 4 = (y-4) + 4 = y$ **(1½)**

Thus, $gof = I_R$, and $fog = I_{[4,\infty)}$.

$\Rightarrow f$ is invertible and its inverse function is g.

$\therefore \quad f^{-1}(y) = g(y) = \sqrt{y-4}$ or $f^{-1}(x) = \sqrt{x-4}$ **(1)**

29. Given function is $f: N \to N$ such that

$$f(x) = \begin{cases} x+1, & \text{if } x \text{ is odd} \\ x-1, & \text{if } x \text{ is even} \end{cases}$$

One-one function Do same as Q. No. 20. **(2)**

Onto function Let $y \in N$ (codomain) be any arbitrary number.

If y is odd, then there exists an even number $y+1 \in N$ (domain) such that

$f(y+1) = (y+1) - 1 = y$.

If y is even, then there exists an odd number $y-1 \in N$ (domain) such that

$f(y-1) = (y-1) + 1 = y$

Thus, every element in N (codomain) has a pre-image in N (domain). **(1)**

Therefore, $f(x)$ is an onto function.

Hence, the function $f(x)$ is bijective. **(1)**

30. Firstly, consider $gof(x) = I_R(x)$, further let $f(x)$ is equal to y and then transform x into y. Finally replace y by x.

Given, $\quad f(x) = 10x + 7$

Also, $\quad gof = fog = I_R$

Now, $\quad gof = I_R \Rightarrow \quad gof(x) = I_R(x)$ **(1)**

$\Rightarrow \quad g[f(x)] = x, \forall x \in R \quad [\because I_R(x) = x, \forall x \in R]$

$\Rightarrow \quad g(10x+7) = x, \forall x \in R$ **(1)**

Let $\quad 10x + 7 = y \quad \Rightarrow \quad 10x = y - 7$

$x = \frac{y-7}{10} \Rightarrow g(y) = \frac{y-7}{10}, \forall y \in R.$

or $\quad g(x) = \frac{x-7}{10}, \forall x \in R$ **(2)**

31. The given function is $f: R \to R$ such that

$$f(x) = 4x^3 + 7$$

To show f is bijective, we have to show that f is one-one and onto.

One-one function Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$.

$\Rightarrow \quad 4x_1^3 + 7 = 4x_2^3 + 7$

$\Rightarrow \quad 4x_1^3 = 4x_2^3 \Rightarrow x_1^3 - x_2^3 = 0$ **(1/2)**

$\Rightarrow \quad (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = 0$

$[\because a^3 - b^3 = (a-b)(a^2 + ab + b^2)]$

$\Rightarrow \quad (x_1 - x_2)\left[\left(x_1 + \frac{x_2}{2}\right)^2 + \frac{3}{4}x_2^2\right] = 0$

$\Rightarrow$ Either $x_1 - x_2 = 0$ ...(i)

or $\left(x_1 + \frac{x_2}{2}\right)^2 + \frac{3}{4}x_2^2 = 0$ ...(ii)

But Eq. (ii) gives complex roots as $x_1, x_2 \in R$.

$\therefore \quad x_1 - x_2 = 0 \Rightarrow x_1 = x_2$

Thus, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2, \ \forall x_1, x_2 \in R$

Therefore, $f(x)$ is a one-one function. **(1)**

Onto function Let $y \in R$ (codomain) be any arbitrary number.

Then, $f(x) = y \Rightarrow 4x^3 + 7 = y \Rightarrow 4x^3 = y - 7$

$\Rightarrow \quad x^3 = \frac{y-7}{4} \Rightarrow x = \left(\frac{y-7}{4}\right)^{1/3}$

which is a real number. $[\because y \in R]$ **(1)**

Thus, for every $y \in R$ (codomain), there exists

$x = \left(\frac{y-7}{4}\right)^{1/3} \in R$ (domain) such that

$$f(x) = f\left[\left(\frac{y-7}{4}\right)^{1/3}\right] = 4\left[\left(\frac{y-7}{4}\right)^{1/3}\right]^3 + 7$$

$$= 4\left(\frac{y-7}{4}\right) + 7 = y - 7 + 7 = y$$

$\Rightarrow f(x)$ is an onto function. **(1/2)**

Since, $f(x)$ is both one-one and onto, so it is a bijective. **(1)**

32. The given relation is $R = \{(a, b) : a, b \in Z$ and $a - b$ is divisible by $5\}$.

To prove R is an equivalence relation, we have to prove R is reflexive, symmetric and transitive.

Reflexive As for any $x \in Z$, we have $x - x = 0$, which is divisible by 5.

$\Rightarrow (x - x)$ is divisible by $5. \Rightarrow (x, x) \in R, \ \forall x \in Z$

Therefore, R is reflexive. **(1)**

Symmetric Let $(x, y) \in R$, where $x, y \in Z$.

$\Rightarrow \quad (x - y)$ is divisible by 5. [by definition of R]

$\Rightarrow x - y = 5A$ for some $A \in Z$. $\Rightarrow y - x = 5(-A)$

$\Rightarrow (y - x)$ is also divisible by 5. $\Rightarrow (y, x) \in R$

Therefore, R is symmetric. **(1)**

Transitive Let $(x, y) \in R$, where $x, y \in Z$.

$\Rightarrow \quad (x - y)$ is divisible by 5.

$\Rightarrow \quad x - y = 5A$ for some $A \in Z$

Again, let $(y, z) \in R$, where $y, z \in Z$.

$\Rightarrow \quad (y - z)$ is divisible by 5.

$\Rightarrow \quad y - z = 5B$ for some $B \in Z$.

Now, $(x - y) + (y - z) = 5A + 5B$

$\Rightarrow \quad x - z = 5(A + B)$

$\Rightarrow \quad (x - z)$ is divisible by 5 for some $(A + B) \in Z$

$\Rightarrow \quad (x, z) \in R$

Therefore, R is transitive. **(1½)**

Thus, R is reflexive, symmetric and transitive. Hence, it is an equivalence relation. **(1/2)**

NOTE If atleast one of the conditions, i.e. reflexive, symmetric and transitive, is not satisfied, then we say that the given relation is not an equivalence relation.

33.

Here, the result is disproved by using some specific examples.

Given relation is

$$S = \{(a, b) : a, b \in R \text{ and } a \le b^3\}.$$

Reflexive As $\frac{1}{2} \le \left(\frac{1}{2}\right)^3$, where $\frac{1}{2} \in R$, is not true.

$\therefore \quad \left(\frac{1}{2}, \frac{1}{2}\right) \notin S$

Thus, S is not reflexive. **(1)**

Symmetric As $-2 \le (3)^3$, where $-2, 3 \in R$, is true but $3 \le (-2)^3$ is not true,

i.e. $(-2, 3) \in S$ but $(3, -2) \notin S$.

Therefore, S is not symmetric. **(1)**

Transitive As $3 \le \left(\frac{3}{2}\right)^3$ and $\frac{3}{2} \le \left(\frac{4}{3}\right)^3$, where $3, \frac{3}{2}, \frac{4}{3} \in R$, are true but $3 \le \left(\frac{4}{3}\right)^3$ is not true.

i.e. $\left(3, \frac{3}{2}\right) \in S$ and $\left(\frac{3}{2}, \frac{4}{3}\right) \in S$ but $\left(3, \frac{4}{3}\right) \notin S$.

Therefore, S is not transitive. **(1½)**

Hence, S is neither reflexive nor symmetric nor transitive. **(1/2)**

34. Given relation is $S = \{(a, b) : |a - b|$ is divisible by 4 and $a, b \in A\}$

and $\quad A = \{x : x \in Z$ and $0 \le x \le 12\}$

Now, A can be written as

$$A = \{0, 1, 2, 3, \ldots, 12\}$$ **(1/2)**

Reflexive As for any $x \in A$, we get $|x - x| = 0$, which is divisible by 4.

$\Rightarrow \quad (x, x) \in S, \forall x \in A$

Therefore, S is reflexive. **(1)**

Symmetric As for any $(x, y) \in S$, we get $|x - y|$ is divisible by 4.

[by using definition of given relation]

$\Rightarrow \quad |x - y| = 4\lambda$, for some $\lambda \in Z$

$\Rightarrow \quad |y - x| = 4\lambda$, for some $\lambda \in Z$

$\Rightarrow \quad (y, x) \in S$

Thus, $\quad (x, y) \in S \Rightarrow (y, x) \in S, \forall x, y \in A$

Therefore, S is symmetric. **(1)**

Transitive For any $(x, y) \in S$ and $(y, z) \in S$, we get $|x - y|$ is divisible by 4 and $|y - z|$ is divisible by 4.

[by using definition of given relation]

$\Rightarrow \quad |x - y| = 4\lambda$ and $|y - z| = 4\mu$,

for some $\lambda, \mu \in Z$.

Now, $x - z = (x - y) + (y - z)$

$= \pm 4\lambda \pm 4\mu$

$= \pm 4(\lambda + \mu)$

$\Rightarrow \quad |x - z|$ is divisible by 4.

$\Rightarrow \quad (x, z) \in S$

Thus, $(x, y) \in S$ and $(y, z) \in S$

$\Rightarrow \quad (x, z) \in S, \forall x, y, z \in A$

Therefore, S is transitive. **(1)**

Since, S is reflexive, symmetric and transitive, so it is an equivalence relation. Now, set of all elements related to 1 is $\{1, 5, 9\}$. **(1/2)**

35. Do same as Q. No. 24.

36. The given function is $f : X \to Y$ and relation on X is $R = \{(a, b) : f(a) = f(b)\}$

Reflexive Since, for every $x \in X$, we have

$$f(x) = f(x)$$

$\Rightarrow \quad (x, x) \in R, \forall x \in X$

Therefore, R is reflexive. **(1)**

Symmetric Let $(x, y) \in R$

Then, $f(x) = f(y) \Rightarrow f(y) = f(x) \Rightarrow (x, y) \in R$

Thus, $(x, y) \in R \Rightarrow (y, x) \in R, \forall x, y \in X$

Therefore, R is symmetric. **(1)**

Transitive Let $x, y, z \in X$ such that

$(x, y) \in R$ and $(y, z) \in R$

Then $\quad f(x) = f(y) \quad \ldots$(i)

and $\quad f(y) = f(z) \quad \ldots$(ii)

From Eqs. (i) and (ii), we get

$$f(x) = f(z)$$

$\Rightarrow \quad (x, z) \in R$

Thus, $\quad (x, y) \in R$ and $(y, z) \in R$

$\Rightarrow \quad (x, z) \in R, \forall x, y, z \in X$

Therefore, R is transitive. **(1½)**

Since, R is reflexive, symmetric and transitive, so it is an equivalence relation. **(1/2)**

37. Given function $f : R \to R$ is such that $f(x) = ax + b; a, b \in R, a \ne 0$.

One-one function Let $x_1, x_2 \in R$ such that

$$f(x_1) = f(x_2)$$

Then, $\quad ax_1 + b = ax_2 + b$

$\Rightarrow \quad ax_1 = ax_2$

$\Rightarrow \quad x_1 = x_2 \quad [\because a \ne 0]$

Thus, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2, \forall x_1, x_2 \in R$ **(1½)**

Therefore, $f(x)$ is a one-one function.

Onto function Let $y \in R$ (codomain) be any arbitrary element. Then, $f(x) = y \Rightarrow ax + b = y$

$\Rightarrow \quad x = \frac{y - b}{a}$

Clearly, x is a real number. $[\because y \in R]$

Thus, for each $y \in R$ (codomain), there exists $x = \frac{y - b}{a} \in R$ (domain) such that

$$f(x) = f\left(\frac{y - b}{a}\right) = a\left(\frac{y - b}{a}\right) + b = y - b + b = y$$

Therefore, $f(x)$ is an onto function. **(1½)**

As $f(x)$ is both one-one and onto, so it is a bijective function. **(1)**

38. Do same as Q. No. 34. **(5)**

The set of all elements related to [2]

$= \{a \in A : |2 - a|$ is divisible by 4$\}$

$= \{2, 6, 10\}$ **(1)**

39. We have, a function $f: R \to R$ defined by

$$f(x) = \frac{x}{x^2+1}, \forall x \in R$$

To show f is neither one-one nor onto.

(i) **One-one** Let $x_1, x_2 \in R$ such that

$$f(x_1) = f(x_2)$$

$$\Rightarrow \frac{x_1}{x_1^2+1} = \frac{x_2}{x_2^2+1}$$

$$\Rightarrow x_1(x_2^2+1) = x_2(x_1^2+1) \quad \textbf{(1)}$$

$$\Rightarrow x_1 x_2^2 + x_1 = x_2 x_1^2 + x_2$$

$$\Rightarrow x_1 x_2(x_2 - x_1) = (x_2 - x_1)$$

$$\Rightarrow (x_2 - x_1)(x_1 x_2 - 1) = 0$$

$$\Rightarrow x_2 = x_1 \text{ or } x_1 x_2 = 1$$

$$\Rightarrow x_1 = x_2 \text{ or } x_1 = \frac{1}{x_2}$$

Here, f is not one-one as if we take.

In particular, $x_1 = 2$ and $x_2 = \frac{1}{2}$, we get

$$f(2) = \frac{2}{4+1} = \frac{2}{5} \text{ and } f\left(\frac{1}{2}\right) = \frac{\frac{1}{2}}{\frac{1}{4}+1} = \frac{\frac{1}{2}}{\frac{5}{4}} = \frac{2}{5}$$

$\because f(2) = f\left(\frac{1}{2}\right)$ but $2 \neq \frac{1}{2}$

$\therefore f$ is not one-one. **(1)**

(ii) **Onto** Let $y \in R$ (codomain) be any arbitrary element.

Consider, $y = f(x)$

$$\therefore y = \frac{x}{x^2+1} \Rightarrow x^2 y + y = x$$

$$\Rightarrow x^2 y - x + y = 0 \quad \textbf{(1)}$$

$\Rightarrow x = \frac{1 \pm \sqrt{1-4y^2}}{2y}$, which does not exist for

$1 - 4y^2 < 0$, i.e for $y > \frac{1}{2}$ and $y < \frac{-1}{2}$

In particular for $y = 1 \in R$ (codomain), there does not exist any $x \in R$ (domain) such that $f(x) = y$.

$\therefore f$ is not onto.

Hence, f is neither one-one nor onto. **(1)**

Now, it is given that $g: R \to R$ defined as

$$g(x) = 2x - 1$$

$$\therefore fog(x) = f(g(x)) = \frac{g(x)}{(g(x))^2+1} = \frac{2x-1}{(2x-1)^2+1}$$

$$= \frac{2x-1}{4x^2+1-4x+1} = \frac{2x-1}{4x^2-4x+2} \quad \textbf{(2)}$$

40. Do same as Q. No. 32.

41. Here, function $f: R_+ \to [-5, \infty)$ is given as

$$f(x) = 9x^2 + 6x - 5$$

One-one function Let $x_1, x_2 \in R_+$ such that

$$f(x_1) = f(x_2)$$

Then, $9x_1^2 + 6x_1 - 5 = 9x_2^2 + 6x_2 - 5$

$$\Rightarrow 9(x_1^2 - x_2^2) + 6(x_1 - x_2) = 0$$

$$\Rightarrow 9(x_1 + x_2)(x_1 - x_2) + 6(x_1 - x_2) = 0$$

$$\Rightarrow (x_1 - x_2)[9(x_1 + x_2) + 6] = 0$$

$$\Rightarrow x_1 - x_2 = 0 \quad \left[\begin{array}{l}\because x_1, x_2 \in R_+ \\ \therefore 9(x_1 + x_2) + 6 \neq 0\end{array}\right]$$

$$\Rightarrow x_1 = x_2, \forall x_1, x_2 \in R_+$$

Therefore, $f(x)$ is one-one function. **(1½)**

Onto function Let y be any arbitrary element of $[-5, \infty)$.

Then, $y = f(x)$

$$\Rightarrow y = 9x^2 + 6x - 5$$

$$\Rightarrow y = (3x+1)^2 - 1 - 5 = (3x+1)^2 - 6$$

$$\Rightarrow (3x+1)^2 = y + 6$$

$$\Rightarrow 3x + 1 = \sqrt{y+6}, \text{ as } y \geq -5 \Rightarrow y + 6 \geq 0$$

$$\Rightarrow x = \frac{\sqrt{y+6}-1}{3}$$

Therefore, f is onto, thereby range $f = [-5, \infty)$ **(1½)**

Let us define $g: [-5, \infty) \to R_+$

as $g(y) = \frac{\sqrt{y+6}-1}{3}$

Now, $(gof)(x) = g[f(x)] = g(9x^2 + 6x - 5)$

$$= g((3x+1)^2 - 6)$$

$$= \frac{\sqrt{(3x+1)^2 - 6 + 6} - 1}{3} = \frac{3x+1-1}{3} = x$$

and $(fog)(y) = f[g(y)] = f\left(\frac{\sqrt{y+6}-1}{3}\right)$

$$= \left[3\left(\frac{\sqrt{y+6}-1}{3}\right) + 1\right]^2 - 6$$

$$= (\sqrt{y+6})^2 - 6$$

$$= y + 6 - 6 = y$$

Therefore, $gof = I_{R+}$ and $fog = I[-5, \infty)$ **(1)**

Hence, f is invertible and the inverse of f is given by

$$f^{-1}(y) = g(y) = \frac{\sqrt{y+6}-1}{3}$$

(i) $\therefore f^{-1}(10) = \frac{\sqrt{10+6}-1}{3} = \frac{\sqrt{16}-1}{3} = \frac{4-1}{3} = 1$ [1]

(ii) If $f^{-1}(y) = \frac{4}{3} \Rightarrow y = f(4/3) = 9(4/3)^2 + 6(4/3) - 5$

$= 16 + 8 - 5 = 19$ [1]

42. Given, $f: R - \left\{-\frac{4}{3}\right\} \to R - \left\{\frac{4}{3}\right\}$

defined by $f(x) = \frac{4x+3}{3x+4}$

Let $x_1, x_2 \in R - \left\{-\frac{4}{3}\right\}$

such that $f(x_1) = f(x_2)$ [1]

$\Rightarrow \frac{4x_1+3}{3x_1+4} = \frac{4x_2+3}{3x_2+4}$

$\Rightarrow (4x_1+3)(3x_2+4) = (3x_1+4)(4x_2+3)$

$\Rightarrow 12x_1x_2 + 16x_1 + 9x_2 + 12 = 12x_1x_2 + 9x_1 + 16x_2 + 12$

$\Rightarrow 7x_1 = 7x_2 \Rightarrow x_1 = x_2 \Rightarrow f$ is one-one. [1]

Let $y \in R - \left\{\frac{4}{3}\right\}$, then $y \neq \frac{4}{3}$

The function f is onto if there exist

$x \in R - \left\{\frac{-4}{3}\right\}$, such that $f(x) = y$ [1]

Now, $f(x) = y \Rightarrow \frac{4x+3}{3x+4} = y \Rightarrow 4x + 3 = y(3x+4)$

$\Rightarrow 4x + 3 = 3xy + 4y \Rightarrow 4x - 3xy = 4y - 3$

$\Rightarrow x(4 - 3y) = 4y - 3$

$\Rightarrow x = \frac{4y-3}{4-3y} \in R - \left\{\frac{-4}{3}\right\} \quad \left(y \neq \frac{4}{3}\right)$

Thus, for any $y \in R - \left\{\frac{4}{3}\right\}$, there exist

$\frac{4y-3}{4-3y} \in R - \left\{\frac{-4}{3}\right\} \Rightarrow f$ is onto.

[1]

Therefore, $f(x)$ is a bijective function.

Since, f is one-one and onto, so f^{-1} exists.

$f^{-1}(y) = \frac{4y-3}{4-3y} \in R - \left\{-\frac{4}{3}\right\}$

$f^{-1}(x) = \frac{4x-3}{4-3x} \Rightarrow f^{-1}(0) = \frac{0-3}{4-0} = \frac{-3}{4}$

and also $f^{-1}(x) = 2 \Rightarrow \frac{4x-3}{4-3x} = 2$ [1]

$\Rightarrow 4x - 3 = 2(4 - 3x) \Rightarrow 4x - 3 = 8 - 6x$

$\Rightarrow 10x = 8 + 3 = 11 \Rightarrow x = \frac{11}{10}$

[1]

43. We have a mapping $f: N \to N$ given by

$f(x) = 9x^2 + 6x - 5$

One-one function Let $x_1, x_2 \in N$, such that $f(x_1) = f(x_2)$

Then, $9x_1^2 + 6x_1 - 5 = 9x_2^2 + 6x_2 - 5$ [1]

$\Rightarrow 9x_1^2 + 6x_1 = 9x_2^2 + 6x_2$

$\Rightarrow 9(x_1^2 - x_2^2) + 6(x_1 - x_2) = 0$

$\Rightarrow 3(x_1 - x_2)(x_1 + x_2) + 2(x_1 - x_2) = 0$

[divide by 3]

$\Rightarrow (x_1 - x_2)(3x_1 + 3x_2 + 2) = 0$

$\Rightarrow x_1 - x_2 = 0$ or $3x_1 + 3x_2 + 2 = 0$

But $3x_1 + 3x_2 + 2 \neq 0$ [$\because x_1, x_2 \in N$] [1]

$\therefore x_1 - x_2 = 0 \Rightarrow x_1 = x_2$

So, f is one-one function.

Onto function Obviously, $f: N \to S$ is an onto function, because S is the range of f. [1]

Thus, $f: N \to S$ is one-one and onto function.

$\Rightarrow f$ is invertible function, so its inverse exists.

Let $f(x) = y$, then $y = 9x^2 + 6x - 5$ [1]

$\Rightarrow y = (3x)^2 + 2\cdot 3x \cdot 1 + 1 - 6 \Rightarrow y = (3x+1)^2 - 6$

$\Rightarrow (3x+1)^2 = y + 6 \Rightarrow 3x + 1 = \sqrt{y+6}$

[taking positive square root as $x \in N$]

$\Rightarrow x = \frac{\sqrt{y+6}-1}{3}$

$\therefore f^{-1}(y) = \frac{\sqrt{y+6}-1}{3}$

or $f^{-1}(x) = \frac{\sqrt{x+6}-1}{3}$, where $x \in S$ [1]

Now, $f^{-1}(43) = \frac{\sqrt{43+6}-1}{3}$

$= \frac{\sqrt{49}-1}{3} = \frac{7-1}{3} = 2$ (1/2)

and $f^{-1}(163) = \frac{\sqrt{163+6}-1}{3} = \frac{\sqrt{169}-1}{3}$

$= \frac{13-1}{3} = \frac{12}{3} = 4$ (1/2)

44. (i) Do same as Q. No. 21.

(ii) We have, $gof(x) = 0, \forall x \in R$

and $fog(x) = \begin{cases} 0, & x > 0 \\ -4x, & x < 0 \end{cases}$

Clearly, $fog(-3) = -4(-3) = 12$

$fog(5) = 0$ and $gof(-2) = 0$.

45. We have, a relation R on $N \times N$ defined by $(a, b)R(c, d)$, if $ad(b + c) = bc(a + d)$.

Reflexive Let $(a, b) \in N \times N$ be any arbitrary element. We have to show $(a, b)\ R\ (a, b)$, i.e. to show $ab(b + a) = ba(a + b)$ which is trivally true as natural numbers are commutative under usual multiplication and addition.

Since, $(a, b) \in N \times N$ was arbitrary, therefore R is reflexive. **(1½)**

Symmetric Let $(a, b), (c, d) \in N \times N$ such that

$(a, b)\ R\ (c, d)$, i.e. $ad(b + c) = bc(a + d)$ ...(i)

To show, $(c, d)\ R\ (a, b)$,

i.e. to show $cb(d + a) = da(c + b)$

From Eq.(i), we have

$$ad(b + c) = bc(a + d) \Rightarrow da(c + b) = cb(d + a)$$

[∵ natural numbers are commutative under usual addition and multiplication]

$\Rightarrow cb(d + a) = da(c + b) \Rightarrow (c, d)\ R\ (a, b)$

Thus, R is symmetric. **(1½)**

Transitive Let $(a, b), (c, d)$ and $(e, f) \in N \times N$ such that $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$.

Now, $(a, b)\ R\ (c, d) \Rightarrow ad(b + c) = bc(a + d)$

$$\Rightarrow \frac{b+c}{bc} = \frac{a+d}{ad} \Rightarrow \frac{1}{b} + \frac{1}{c} = \frac{1}{a} + \frac{1}{d} \quad \text{...(ii)}$$

and $(c, d)\ R\ (e, f) \Rightarrow cf(d + e) = de(c + f)$

$$\Rightarrow \frac{d+e}{de} = \frac{c+f}{cf}$$

$$\Rightarrow \frac{1}{d} + \frac{1}{e} = \frac{1}{c} + \frac{1}{f} \quad \text{...(iii)} \ \textbf{(1)}$$

On adding Eqs. (ii) and (iii), we get

$$\left(\frac{1}{b} + \frac{1}{c}\right) + \left(\frac{1}{d} + \frac{1}{e}\right) = \left(\frac{1}{a} + \frac{1}{d}\right) + \left(\frac{1}{c} + \frac{1}{f}\right)$$

$$\Rightarrow \frac{1}{b} + \frac{1}{e} = \frac{1}{a} + \frac{1}{f} \Rightarrow \frac{e+b}{be} = \frac{f+a}{af}$$

$$\Rightarrow af(e + b) = be(f + a)$$

$$\Rightarrow af(b + e) = be(a + f) \Rightarrow (a, b)\ R\ (e, f)$$

$\Rightarrow R$ is transitive. **(1)**

Thus, R is reflexive, symmetric and transitive, hence R is an equivalence relation. **(1)**

46. Do same as Q. No. 41.

47. Do same as Q. No. 43. $\left[\textbf{Ans. } f^{-1}(x) = \frac{\sqrt{x-6}-3}{2}\right]$

48. (i) Do same as Q. No. 34.

(ii) Clearly, $[1] = \{1, 3, 5\}$ $\quad [2] = \{2, 4\}$

$[3] = \{1, 3, 5\}$ $\quad [4] = \{2, 4\}$

and $[5] = \{1, 3, 5\}$

Thus, $[1] = [3] = [5] = \{1, 3, 5\}$ and $[2] = [4] = \{2, 4\}$

Solutions

1. (c) R is reflexive as $(3, 3), (6, 6), (9, 9), (12, 12) \in R$. R is not symmetric as $(6, 12) \in R$ but $(12, 6) \notin R$. R is transitive as the only pair which needs verification is $(3, 6)$ and $(6, 12) \in R$.

$$\Rightarrow \quad (3, 12) \in R$$

2. (d) **Reflexive** As $20 \in N$ but $(20, 20) \notin R$.
So, it is not reflexive.
Symmetric As $(20, 30) \in R$ but $(30, 20) \notin R$.
So, it is not symmetric.
Transitive As $(20, 30) \in R$, $(30, 50) \in R$ but $(20, 50) \notin R$.
So, it is not transitive.

3. (a) If A and B are equivalence relations, then $A \cap B$ is also an equivalence relation.

4. (c) In the given options, only option (c) satisfies the condition of a function.

5. (d) Given, $A = \{1, 3, 5, 7\}$
and $B = \{1, 2, 3, 4, 5, 6, 7, 8\}$
Here, $n(A) = 4$ and $n(B) = 8$
$\therefore$ Number of one-one function from A into B
$$= {}^8P_4 = 8 \cdot 7 \cdot 6 \cdot 5 = 1680$$

6. (d) Given function is
$$f(x) = x^2 + bx + c.$$
It is a quadratic equation in x.
So, we will get a parabola either downward or upward.
Hence, it is a many-one mapping and not onto mapping.
Hence, it is neither one-one nor onto mapping.

7. (c) $f(1) = 3, f(2) = 4, f(3) = 5, f(4) = 6$
$\Rightarrow 1 \in B, 2 \in B$ do not have any pre-image in A.
$\Rightarrow f$ is one-one and into.

8. (d) Since, $f(n) = \begin{cases} n^2, & \text{if } n \text{ is odd.} \\ 2n + 1, & \text{if } n \text{ is even.} \end{cases}$

$\therefore$
$$f(1) = 1^2 = 1,$$
$$f(2) = 2(2) + 1 = 5$$
$$f(3) = 3^2 = 9,$$
$$f(4) = 2(4) + 1 = 9$$
$\because$
$$f(3) = f(4)$$
So, f is not injective.
Also, f is not surjective as some element of N (codomain) is not the image of any element of N.

9. (*d*) Given, $f(x) = \sqrt{\cos x}$, i.e. $\cos x \geq 0$

But $-1 \leq \cos x \leq 1$

$\therefore \quad 0 \leq \cos x \leq 1$

i.e. x lies in Ist or IVth quadrant.

$\Rightarrow \quad 0 \leq x \leq \frac{\pi}{2}$ or $\frac{3\pi}{2} \leq x \leq 2\pi$

$\therefore \quad x \in \left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$

Also, $\cos(-x) = \cos x$

$\therefore \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is also the domain of the function.

10. (*d*) Given, $f(x) = x^2 + 2x + 2 = x^2 + 2x + 1 + 1$

$= (x+1)^2 + 1 \geq 1$

So, the range of $f(x)$ is $[1, \infty)$.

11. (*c*) Let $y = \frac{x}{1+x^2} \Rightarrow x^2y - x + y = 0$

For x to be real, $1 - 4y^2 \geq 0$

$\Rightarrow \quad (1-2y)(1+2y) \geq 0$

$\Rightarrow \quad \left(\frac{1}{2} - y\right)\left(\frac{1}{2} + y\right) \geq 0$

$\Rightarrow \quad -\frac{1}{2} \leq y \leq \frac{1}{2}$

$\therefore \quad y = f(x) \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

12. (*d*) Given, $f(x) = \sqrt{x}$ and $g(x) = 2x - 3$

$\therefore \quad fog(x) = f\{g(x)\} = f(2x-3) = \sqrt{2x-3}$

For domain of $fog(x)$,

$2x - 3 \geq 0$

$\Rightarrow \quad 2x \geq 3$

$\Rightarrow \quad x \geq \frac{3}{2}$

$\therefore \quad x \in \left[\frac{3}{2}, \infty\right)$

13. (*a*) Given, $f(x) = \frac{x+2}{3x-1}$

$\therefore \quad f\{f(x)\} = f\left(\frac{x+2}{3x-1}\right)$

$= \frac{\frac{x+2}{3x-1} + 2}{3\left(\frac{x+2}{3x-1}\right) - 1}$

$= \frac{x+2+6x-2}{3x+6-(3x-1)} = \frac{7x}{7} = x$

14. (*a*) Let $y = f(x) = x^3$, then

$x = y^{1/3}$

$\Rightarrow \quad f^{-1}(x) = x^{1/3}$

$\therefore \quad f^{-1}(8) = (8)^{1/3} = 2$