DCAM classes
Dynamic Classes for Academic Mastery

Solutions

1. Given, $f(x)=\begin{cases}\dfrac{(x+3)^2-36}{x-3}, & x\neq 3\\ k, & x=3\end{cases}$

Let $f(x)$ is continuous at $x=3$

Then, we have $\lim\limits_{x\to 3} f(x)=f(3)$

$\Rightarrow \lim\limits_{x\to 3}\dfrac{(x+3)^2-36}{x-3}=k$

$\Rightarrow \lim\limits_{x\to 3}\dfrac{(x+3)^2-6^2}{x-3}=k$

$\Rightarrow \lim\limits_{x\to 3}\dfrac{(x+3-6)(x+3+6)}{x-3}=k$ $[\because a^2-b^2=(a-b)(a+b)]$

$\Rightarrow \lim\limits_{x\to 3}\dfrac{(x-3)(x+9)}{(x-3)}=k$

$\Rightarrow \lim\limits_{x\to 3}(x+9)=k$

$\Rightarrow 3+9=k \Rightarrow k=12$ **(1)**

2. Let $f(x)=\begin{cases}\dfrac{kx}{|x|}, & \text{if } x<0\\ 3, & \text{if } x\geq 0\end{cases}$ is continuous at $x=0$

Then, $\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^-} f(x)=f(0)$

$\Rightarrow \lim\limits_{h\to 0} f(0+h)=\lim\limits_{h\to 0} f(0-h)=f(0)$

$\Rightarrow 3=\lim\limits_{h\to 0}\dfrac{k(-h)}{|-h|}=3$

$\Rightarrow \lim\limits_{h\to 0}\left(\dfrac{-kh}{h}\right)=3 \Rightarrow \lim\limits_{h\to 0}(-k)=3$

$\therefore k=-3$ **(1)**

3. Let $f(x)=\begin{cases}\dfrac{1-\sin^3 x}{3\cos^2 x}, & \text{if } x<\dfrac{\pi}{2}\\ p, & \text{if } x=\dfrac{\pi}{2}\\ \dfrac{q(1-\sin x)}{(\pi-2x)^2}, & \text{if } x>\dfrac{\pi}{2}\end{cases}$

is continuous at $x=\dfrac{\pi}{2}$.

Then, $(\text{LHL})_{x=\frac{\pi}{2}}=(\text{RHL})_{x=\frac{\pi}{2}}=f\left(\dfrac{\pi}{2}\right)$ ...(i)

Now, LHL $=\lim\limits_{x\to\frac{\pi}{2}^-} f(x)=\lim\limits_{h\to 0} f\left(\dfrac{\pi}{2}-h\right)$

$\left[\text{put } x=\dfrac{\pi}{2}-h; \text{ when } x\to\dfrac{\pi^-}{2}, \text{ then } h\to 0\right]$

$=\lim\limits_{h\to 0}\dfrac{1-\sin^3\left(\dfrac{\pi}{2}-h\right)}{3\cos^2\left(\dfrac{\pi}{2}-h\right)}=\lim\limits_{h\to 0}\dfrac{1-\cos^3 h}{3\sin^2 h}$ **(1/2)**

$\left[\because \cos\left(\dfrac{\pi}{2}-\theta\right)=\sin\theta, \sin\left(\dfrac{\pi}{2}-\theta\right)=\cos\theta\right]$

$=\lim\limits_{h\to 0}\dfrac{(1-\cos h)(1^2+\cos^2 h+1\times\cos h)}{3(1-\cos^2 h)}$

$=\lim\limits_{h\to 0}\dfrac{(1-\cos h)(1+\cos^2 h+\cos h)}{3(1-\cos h)(1+\cos h)}$

$=\lim\limits_{h\to 0}\dfrac{(1+\cos^2 h+\cos h)}{3(1+\cos h)}$

$=\dfrac{1+\cos^2 0+\cos 0}{3(1+\cos 0)}=\dfrac{1+1+1}{3(1+1)}=\dfrac{3}{3\times 2}=\dfrac{1}{2}$

...(ii) **(1)**

and RHL $=\lim\limits_{x\to\frac{\pi}{2}^+} f(x)=\lim\limits_{h\to 0} f\left(\dfrac{\pi}{2}+h\right)$

$\left[\text{put } x=\dfrac{\pi}{2}+h; \text{ when } x\to\dfrac{\pi^+}{2}, \text{ then } h\to 0\right]$

$=\lim\limits_{h\to 0}\dfrac{q\left[1-\sin\left(\dfrac{\pi}{2}+h\right)\right]}{\left[\pi-2\left(\dfrac{\pi}{2}+h\right)\right]^2}$ **(1/2)**

$=\lim\limits_{h\to 0}\dfrac{q(1-\cos h)}{(\pi-\pi-2h)^2}=\lim\limits_{h\to 0}\dfrac{q(1-\cos h)}{4h^2}$

$=\lim\limits_{h\to 0}\dfrac{q\left(2\sin^2\dfrac{h}{2}\right)}{4h^2}$ $\left[\because \cos x=1-2\sin^2\dfrac{x}{2}\right]$

$$= \frac{q}{8} \lim_{h \to 0} \left[\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \right]^2$$

$$= \frac{q}{8} \times 1 = \frac{q}{8} \qquad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right] \quad ...(iii)\ (1)$$

On substituting the values from Eqs. (ii) and (iii) to Eq. (i), we get

$$\frac{1}{2} = \frac{q}{8} = f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \quad \frac{1}{2} = \frac{q}{8} = p \qquad \left[\because f\left(\frac{\pi}{2}\right) = p \text{ (given)}\right]$$

$$\Rightarrow \quad \frac{1}{2} = \frac{q}{8} \text{ and } \frac{1}{2} = p$$

$$\therefore \quad q = 4 \text{ and } p = \frac{1}{2} \qquad (1)$$

4. Given,

$$f(x) = \begin{cases} \dfrac{\sin (a+1)x + 2\sin x}{x}, & x < 0 \\ 2, & x = 0 \\ \dfrac{\sqrt{1+bx} - 1}{x}, & x > 0 \end{cases}$$

is continuous at $x = 0$.

$$\therefore \quad (\text{LHL})_{x=0} = (\text{RHL})_{x=0} = f(0) \qquad ...(i)\ (1/2)$$

Now, LHL $= \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h)$

[put $x = 0 - h$; when $x \to 0^-$, then $h \to 0$]

$$= \lim_{h \to 0} \frac{\sin [(a+1)(0-h)] + 2\sin (0-h)}{(0-h)}$$

$$= \lim_{h \to 0} \frac{-\sin [(a+1)h] - 2\sin h}{-h}$$

$[\because \sin(-\theta) = -\sin\theta]$

$$= \lim_{h \to 0} \frac{\sin (a+1)h + 2\sin h}{h}$$

$$= \lim_{h \to 0} \frac{\sin (a+1)h}{h} + \lim_{h \to 0} 2\frac{\sin h}{h}$$

$$= \lim_{h \to 0} \frac{\sin (a+1)h}{(a+1)h} \times (a+1) + 2 \lim_{h \to 0} \frac{\sin h}{h}$$

$$= 1 \times (a+1) + 2 \times 1 \qquad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

$$= a + 1 + 2 = a + 3 \qquad ...(ii)\ (1)$$

and RHL $= \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h)$

[put $x = 0 + h$; when $x \to 0^+$, then $h \to 0$] (1/2)

$$= \lim_{h \to 0} \frac{\sqrt{1 + b(0+h)} - 1}{0+h}$$

$$= \lim_{h \to 0} \frac{\sqrt{1+bh} - 1}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{1+bh} - 1}{h} \times \frac{\sqrt{1+bh} + 1}{\sqrt{1+bh} + 1}$$

[multiplying numerator and denominator by $\sqrt{1+bh} + 1$]

$$= \lim_{h \to 0} \frac{(1+bh) - 1}{h(\sqrt{1+bh} + 1)} = \lim_{h \to 0} \frac{bh}{h(\sqrt{1+bh} + 1)}$$

$$= \lim_{h \to 0} \frac{b}{(\sqrt{1+bh} + 1)} = \frac{b}{\sqrt{1+0} + 1} = \frac{b}{2} \qquad ...(iii)\ (1)$$

From Eqs. (i), (ii) and (iii) we get

$$a + 3 = \frac{b}{2} = 2 \qquad [\because f(0) = 2]$$

$$\Rightarrow \quad a + 3 = 2 \text{ and } \frac{b}{2} = 2$$

$$\therefore \quad a = -1 \text{ and } b = 4 \qquad (1)$$

5. Let $f(x) = \begin{cases} \left(\dfrac{1 - \cos 4x}{8x^2}\right), & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$

is continuous at $x = 0$.

Then, $(\text{LHL})_{x=0} = (\text{RHL})_{x=0} = f(0)$ ...(i)

Now, LHL $= \lim_{x \to 0^-} f(x)$ (1/2)

$$= \lim_{x \to 0^-} \frac{1 - \cos 4x}{8x^2}$$

$$= \lim_{h \to 0} \frac{1 - \cos 4(0-h)}{8(0-h)^2}$$

[put $x = 0 - h$; when $x \to 0^-$, then $h \to 0$] (1)

$$= \lim_{h \to 0} \frac{1 - \cos 4h}{8h^2} \qquad [\because \cos(-\theta) = \cos\theta]$$

$$= \lim_{h \to 0} \frac{2\sin^2 2h}{8h^2} \qquad [\because 1 - \cos 2\theta = 2\sin^2\theta]$$

$$= \lim_{h \to 0} \frac{\sin^2 2h}{4h^2} = \lim_{h \to 0} \left(\frac{\sin 2h}{2h}\right)^2 = 1 \qquad (1)$$

$$\left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

On substituting this value in Eq. (i), we get

$$1 = f(0) \Rightarrow 1 = k \qquad [\because f(0) = k, \text{(given)}] \ (1)$$

Hence, for $k = 1$, the given function $f(x)$ is continuous at $x = 0$.

Alternate Method (1/2)

Let $f(x) = \begin{cases} \left(\dfrac{1 - \cos 4x}{8x^2}\right), & \text{if } x = 0 \\ k, & \text{if } x = 0 \end{cases}$ is

continuous at $x = 0$.

(1/2)

Here, $f(0) = k$ (1/2)

and $\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{1-\cos 4x}{8x^2} = \lim_{x\to 0}\frac{2\sin^2 2x}{8x^2}$

$= \lim_{x\to 0}\left(\frac{\sin 2x}{2x}\right)^2 = 1$ (2)

$\because f(x)$ is continuous at $x = 0$.

$\therefore \quad \lim_{x\to 0} f(x) = f(0) \Rightarrow 1 = k \Rightarrow k = 1$ (1)

6. Given, $f(x) = \begin{cases} \frac{1-\cos 4x}{x^2}, & \text{when } x < 0 \\ a, & \text{when } x = 0 \\ \frac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}, & \text{when } x > 0 \end{cases}$

Since, $f(x)$ is continuous at $x = 0$.

$\therefore \quad (\text{LHL})_{x=0} = (\text{RHL})_{x=0} = f(0)$ ...(i)

Now, $(\text{LHL})_{x=0} = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\frac{1-\cos 4x}{x^2}$ (1)

$= \lim_{h\to 0}\frac{1-\cos 4(0-h)}{(0-h)^2}$

[put $x = 0-h$; when $x \to 0^-$, then $h \to 0$]

$= \lim_{h\to 0}\frac{1-\cos 4h}{h^2}$ $[\because \cos(-\theta) = \cos\theta]$ (1)

$= \lim_{h\to 0}\frac{2\sin^2 2h}{h^2}$ $[\because 1-\cos 2\theta = 2\sin^2\theta]$

$= 2\lim_{h\to 0}\left(\frac{\sin 2h}{2h}\right)^2 \times 4$ (1)

$= 2\times(1)^2\times 4$ $\left[\because \lim_{x\to 0}\frac{\sin x}{x} = 1\right]$

$= 8$

Now, from Eq. (i), we have

$(\text{LHL})_{x=0} = f(0)$

$\Rightarrow \quad 8 = a$ $[\because f(0) = a \text{ (given)}]$

$\therefore \quad a = 8$ (1)

7. Let $f(x) = \begin{cases} \frac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & \text{if } -1 \le x < 0 \\ \frac{2x+1}{x-1}, & \text{if } 0 \le x < 1 \end{cases}$

is continuous at $x = 0$.

Now, $\quad f(0) = \frac{2\cdot 0+1}{0-1} = \frac{1}{-1} = -1$ (1)

and $\quad \text{LHL} = \lim_{h\to 0} f(0-h)$

$= \lim_{h\to 0}\frac{\sqrt{1-kh}-\sqrt{1+kh}}{-h}$

$= \lim_{h\to 0}\frac{\sqrt{1-kh}-\sqrt{1+kh}}{-h}\times\frac{(\sqrt{1-kh}+\sqrt{1+kh})}{(\sqrt{1-kh}+\sqrt{1+kh})}$ (1)

$= \lim_{h\to 0}\frac{(1-kh)-(1+kh)}{-h(\sqrt{1-kh}+\sqrt{1+kh})}$

$[\because (a+b)(a-b) = a^2-b^2]$

$= \lim_{h\to 0}\frac{-2kh}{-h(\sqrt{1-kh}+\sqrt{1+kh})}$ (1)

$= \lim_{h\to 0}\frac{2k}{\sqrt{1-kh}+\sqrt{1+kh}} = \frac{2k}{1+1} = \frac{2k}{2} = k$

Since, $f(x)$ is continuous at $x = 0$.

$\therefore \quad f(0) = \text{LHL} \Rightarrow -1 = k$

$\Rightarrow \quad k = -1$ (1)

8. Let $f(x) = \begin{cases} \frac{x^3+x^2-16x+20}{(x-2)^2}; & x \ne 2 \\ k; & x = 2 \end{cases}$

is continuous at $x = 2$. (1/2)

Now, we have $f(2) = k$ (1/2)

and $\lim_{x\to 2} f(x) = \lim_{x\to 2}\frac{x^3+x^2-16x+20}{(x-2)^2}$

$= \lim_{x\to 2}\frac{(x-2)(x^2+3x-10)}{(x-2)^2}$ (1)

$= \lim_{x\to 2}\frac{(x-2)(x+5)(x-2)}{(x-2)^2}$

$= \lim_{x\to 2}(x+5) = 2+5 = 7$ (1)

Since, $f(x)$ is continuous at $x = 2$

$\therefore \lim_{x\to 2} f(x) = f(2) \Rightarrow 7 = k \Rightarrow k = 7$ (1)

9. Let $f(x) = \begin{cases} \frac{k\cos x}{\pi-2x}, & \text{if } x \ne \frac{\pi}{2} \\ 3, & \text{if } x = \frac{\pi}{2} \end{cases}$

is continuous at $x = \pi/2$.

Then, at $x = \frac{\pi}{2}$, $\text{LHL} = \text{RHL} = f\left(\frac{\pi}{2}\right)$ ...(i) (1)

Now, $\text{LHL} = \lim_{x\to\frac{\pi}{2}^-} f(x) = \lim_{x\to\frac{\pi}{2}^-}\frac{k\cos x}{\pi-2x}$

$\Rightarrow \quad \text{LHL} = \lim_{h\to 0}\frac{k\cos\left(\frac{\pi}{2}-h\right)}{\pi-2\left(\frac{\pi}{2}-h\right)}$

$\left[\text{put } x = \frac{\pi}{2}-h;\text{ when } x \to \frac{\pi}{2}^-, \text{ then } h \to 0\right]$

$$= \lim_{h\to 0} \frac{k \sin h}{\pi - \pi + 2h} \left[\because \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta\right]$$

$$= \lim_{h\to 0} \frac{k \sin h}{2h} = \frac{k}{2} \lim_{h\to 0} \frac{\sin h}{h}$$

$$\Rightarrow \text{LHL} = \frac{k}{2} \qquad \left[\because \lim_{h\to 0} \frac{\sin h}{h} = 1\right] \quad \textbf{(1½)}$$

Also, from the given function, we get

$$f\left(\frac{\pi}{2}\right) = 3 \qquad \textbf{(1/2)}$$

Now, from Eq. (i), we have

$$\text{LHL} = f\left(\frac{\pi}{2}\right) \Rightarrow \frac{k}{2} = 3$$

$\therefore$ $k = 6$ **(1)**

10. Let $f(x) = \begin{cases} a\sin\frac{\pi}{2}(x+1), & x \le 0 \\ \frac{\tan x - \sin x}{x^3}, & x > 0 \end{cases}$

is continuous at $x = 0$.

Then, $\text{LHL} = \text{RHL} = f(0)$ ...(i) **(1/2)**

Now, $\text{LHL} = \lim_{x\to 0^-} a \sin\frac{\pi}{2}(x+1)$

$\Rightarrow$ $\text{LHL} = \lim_{h\to 0} a \sin\frac{\pi}{2}(-h+1)$

[put $x = 0 - h$; when $x \to 0^-$, then $h \to 0$]

$$= a\sin\frac{\pi}{2} = a \qquad \text{...(ii)}$$

$$f(0) = a\sin\frac{\pi}{2} = a \qquad \text{...(iii)} \ \textbf{(1)}$$

Now, we need to evaluate RHL at $x = a$.

[$\because$ LHL $= f(0) = a$ and from this, we can't find the value of a]

Here, $\text{RHL} = \lim_{x\to 0^+} \frac{\tan x - \sin x}{x^3}$

$\Rightarrow$ $\text{RHL} = \lim_{h\to 0} \frac{\tan h - \sin h}{h^3}$

[put $x = 0 + h = h$; when $x \to 0^+$, then $h \to 0$]

$$= \lim_{h\to 0} \frac{\frac{\sin h}{\cos h} - \sin h}{h^3}$$

$$= \lim_{h\to 0} \frac{\sin h - \sin h \cos h}{h^3 \cos h}$$

$$= \lim_{h\to 0} \frac{\sin h(1-\cos h)}{h^3 \cos h} \qquad \textbf{(1/2)}$$

$$= \lim_{h\to 0} \frac{\sin h}{h} \cdot \lim_{h\to 0} \frac{1-\cos h}{h^2} \cdot \lim_{h\to 0} \frac{1}{\cos h}$$

$$= 1 \times \lim_{h\to 0} \frac{1-\cos h}{h^2} \times 1$$

$$\left[\because \lim_{h\to 0} \frac{\sin h}{h} = 1 \text{ and } \lim_{h\to 0} \frac{1}{\cos h} = \frac{1}{\cos 0} = \frac{1}{1} = 1\right]$$

$$= \lim_{h\to 0} \frac{1-\cos h}{h^2} \qquad \textbf{(1/2)}$$

$$= \lim_{h\to 0} \frac{2\sin^2\frac{h}{2}}{h^2} \qquad \left[\because 1 - \cos x = 2\sin^2\frac{x}{2}\right]$$

$$= \lim_{h\to 0} \frac{2\times\sin^2\frac{h}{2}}{\left(\frac{h^2}{4}\times 4\right)}$$

$$= \lim_{h\to 0} \frac{2}{4} \times \lim_{h\to 0} \frac{\sin^2\frac{h}{2}}{\frac{h^2}{4}}$$

$$= \frac{1}{2} \times \lim_{h\to 0} \left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2 = \frac{1}{2} \times 1$$

$$\left[\because \lim_{x\to 0} \frac{\sin x}{x} = 1\right] \quad \textbf{(1)}$$

$$\therefore \ \text{RHL} = \frac{1}{2} \qquad \text{...(iv)}$$

On substituting the values from Eqs. (ii), (iii) and (iv) to Eq. (i), we get

$$a = \frac{1}{2} = a \Rightarrow a = \frac{1}{2} \qquad \textbf{(1/2)}$$

11. Given, $f(x) = \begin{cases} 3ax + b, & \text{if } x > 1 \\ 11, & \text{if } x = 1 \\ 5ax - 2b, & \text{if } x < 1 \end{cases}$

is continuous at $x = 1$.

$\therefore$ $\text{LHL} = \text{RHL} = f(1)$ ...(i)

Now, $\text{LHL} = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (5ax - 2b)$ **(1/2)**

$$= \lim_{h\to 0} [5a(1-h) - 2b]$$

[put $x = 1 - h$; when $x \to 1^-$, then $h \to 0$]

$$= \lim_{h\to 0} (5a - 5ah - 2b) = 5a - 2b \qquad \textbf{(1)}$$

and $\text{RHL} = \lim_{x\to 1^+} (3ax + b) = \lim_{h\to 0} [3a(1+h) + b]$

[put $x = 1 + h$; when $x \to 1^+$, then $h \to 0$]

$$= \lim_{h\to 0} (3a + 3ah + b) = 3a + b$$

Also, given that $f(1) = 11$ **(1)**

On substituting these values in Eq. (i), we get

$$5a-2b=3a+b=11$$

$\Rightarrow$ $3a+b=11$ ...(ii)

and $5a-2b=11$ ...(iii) (1/2)

On subtracting $3\times$ Eq. (iii) from $5\times$ Eq. (ii), we get

$$15a+5b-15a+6b=55-33$$

$\Rightarrow$ $11b=22 \Rightarrow b=2$

On putting the value of b in Eq. (ii), we get

$$3a+2=11 \Rightarrow 3a=9 \Rightarrow a=3$$

Hence, $a=3$ and $b=2$ (1)

12. Let $f(x)=\begin{cases}5, & x\le 2\\ ax+b, & 2<x<10\\ 21, & x\ge 10\end{cases}$

is a continuous function. So, it is continuous at $x=2$ and at $x=10$.

$\therefore$ By definition,

$(\text{LHL})_{x=2}=(\text{RHL})_{x=2}=f(2)$ ...(i)

and $(\text{LHL})_{x=10}=(\text{RHL})_{x=10}=f(10)$ ...(ii) (1)

Now, let us calculate LHL and RHL at $x=2$.

$$\text{LHL}=\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-} 5=5$$

and $\text{RHL}=\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}(ax+b)$

$=\lim_{h\to 0}\{a(2+h)+b\}=\lim_{h\to 0}(2a+ah+b)$

[put $x=2+h$; when $x\to 2^+$, then $h\to 0$]

$=2a+b$

From Eq. (i), we have

LHL = RHL

$\Rightarrow$ $2a+b=5$ ...(iii) (1)

Now, we have to find LHL and RHL at $x=10$.

$\text{LHL}=\lim_{x\to 10^-} f(x)=\lim_{x\to 10^-}(ax+b)$

$=\lim_{h\to 0}[a(10-h)+b]$

[put $x=10-h$; when $x\to 10^-$, then $h\to 0$]

$=\lim_{h\to 0}(10a-ah+b)$

$\Rightarrow$ $\text{LHL}=10a+b$

and $\text{RHL}=\lim_{x\to 10^+} f(x)=\lim_{x\to 10^+} 21=21$

Now, from Eq. (ii), we have

LHL = RHL

$\Rightarrow$ $10a+b=21$ ...(iv) (1)

On subtracting Eq. (iv) from Eq. (iii), we get

$$-8a=-16 \Rightarrow a=2$$

On putting $a=2$ in Eq. (iv), we get

$$20+b=21 \Rightarrow b=1$$

Hence, $a=2$ and $b=1$. (1)

13. Let $f(x)=\begin{cases}ax+1, & \text{if } x\le 3\\ bx+3, & \text{if } x>3\end{cases}$

is continuous at point $x=3$.

Then, $\text{LHL}=\text{RHL}=f(3)$. ...(i) (1)

Now, $\text{LHL}=\lim_{x\to 3^-} f(x)=\lim_{x\to 3^-}(ax+1)$

$=\lim_{h\to 0}[a(3-h)+1]$

[put $x=3-h$; when $x\to 3^-$, then $h\to 0$]

$=\lim_{h\to 0}(3a-ah+1)$

$\Rightarrow$ $\text{LHL}=3a+1$ (1)

and $\text{RHL}=\lim_{x\to 3^+} f(x)=\lim_{x\to 3^+}(bx+3)$

$=\lim_{h\to 0}[b(3+h)+3]$

[put $x=3+h$; when $x\to 3^+$, then $h\to 0$]

$=\lim_{h\to 0}(3b+bh+3)$

$\Rightarrow$ $\text{RHL}=3b+3$ (1)

From Eq. (i), we have

$$\text{LHL}=\text{RHL} \Rightarrow 3a+1=3b+3$$

Then, $3a-3b=2$, which is the required relation between a and b. (1)

14. Let $f(x)=\begin{cases}kx+1, & \text{if } x\le \pi\\ \cos x, & \text{if } x>\pi\end{cases}$

is continuous at $x=\pi$.

Then, $\text{LHL}=\text{RHL}=f(\pi)$ ...(i) (1)

Now, $\text{LHL}=\lim_{x\to \pi^-} f(x)=\lim_{x\to \pi^-}(kx+1)$

$=\lim_{h\to 0}[k(\pi-h)+1]$

[put $x=\pi-h$; when $x\to \pi^-$, then $h\to 0$]

$=\lim_{h\to 0}(k\pi-kh+1)=k\pi+1$ (1)

and $\text{RHL}=\lim_{x\to \pi^+} f(x)=\lim_{x\to \pi^+}\cos x$

$=\lim_{h\to 0}\cos(\pi+h)$

[put $x=\pi+h$; when $x\to \pi^+$, then $h\to 0$]

$=\cos \pi$ (1)

$=-1$ [$\because \cos \pi=-1$]

Now, from Eq. (i), we have

$$\text{LHL} = \text{RHL}$$

$$\Rightarrow \quad k\pi + 1 = -1 \Rightarrow k\pi = -2$$

$$\therefore \quad k = -\frac{2}{\pi} \quad \textbf{(1)}$$

15. Let $f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{if } x \le 0 \\ 4x + 1, & \text{if } x > 0 \end{cases}$

is continuous at $x = 0$.

Then, $(\text{LHL})_{x=0} = (\text{RHL})_{x=0} = f(0)$ ...(i) **(1)**

Now, $f(0) = \lambda[0 - 0] = 0$,

$$\text{LHL} = \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h)$$

$$= \lambda \lim_{h \to 0} [(0 - h)^2 - 2(0 - h)]$$

$$= \lambda \times 0 = 0 \quad \textbf{(1)}$$

and $\text{RHL} = \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h)$

$$= \lim_{h \to 0} 4(0 + h) + 1 = 1 \quad \textbf{(1)}$$

$\because$ LHL $\ne$ RHL, which is a contradiction to Eq. (i).

$\therefore$ There is no value of λ for which $f(x)$ is continuous at $x = 0$. **(1)**

16.

> Here, we find LHL, RHL and $f\left(\frac{1}{2}\right)$. If LHL $=$ RHL $= f\left(\frac{1}{2}\right)$, then we say that $f(x)$ is continuous at $x = \frac{1}{2}$, otherwise $f(x)$ is discontinuous at $x = \frac{1}{2}$.

Given function is

$$f(x) = \begin{cases} \frac{1}{2} + x, & 0 \le x < \frac{1}{2} \\ 1, & x = \frac{1}{2} \\ \frac{3}{2} + x, & \frac{1}{2} < x \le 1 \end{cases}$$

We have to check continuity of $f(x)$ at $x = \frac{1}{2}$. **(1)**

Now, $\text{LHL} = \lim_{x \to \frac{1}{2}^-} f(x) = \lim_{x \to \frac{1}{2}^-} \left(\frac{1}{2} + x\right)$

$$= \lim_{h \to 0} \left(\frac{1}{2} + \frac{1}{2} - h\right)$$

$\left[\text{put } x = \frac{1}{2} - h; \text{ when } x \to \frac{1^-}{2}, \text{ then } h \to 0\right]$

$$= \lim_{h \to 0} (1 - h) = 1 \quad \textbf{(1)}$$

and $\text{RHL} = \lim_{x \to \frac{1}{2}^+} f(x) = \lim_{x \to \frac{1}{2}^+} \left(\frac{3}{2} + x\right)$

$$= \lim_{h \to 0} \left(\frac{3}{2} + \frac{1}{2} + h\right)$$

$\left[\text{put } x = \frac{1}{2} + h; \text{ when } x \to \frac{1^+}{2}, \text{ then } h \to 0\right]$

$$= \lim_{h \to 0} (2 + h) = 2 \quad \textbf{(1)}$$

$\because$ LHL $\ne$ RHL at $x = 1/2$.

$\therefore f(x)$ is discontinuous at $x = \frac{1}{2}$. **(1)**

17. Given, $f(x) = \begin{cases} 2x - 1, & x < 2 \\ a, & x = 2 \\ x + 1, & x > 2 \end{cases}$ is continuous at $x = 2$

$\therefore \quad (\text{LHL})_{x=2} = (\text{RHL})_{x=2} = f(2)$ ...(i) **(1)**

Now, $f(2) = a$

and $\text{LHL} = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2x - 1)$

$$= \lim_{h \to 0} [2(2 - h) - 1] = 3 \quad \textbf{(1)}$$

[put $x = 2 - h$; when $x \to 2^-$, then $h \to 0$]

From Eq. (i), we have

$$\text{LHL} = f(2) \Rightarrow a = 3 \quad \textbf{(1/2)}$$

Now, let us check the continuity at $x = 3$.

Consider, $\lim_{x \to 3} f(x) = \lim_{x \to 3} (x + 1)$

[$\because f(x) = x + 1$ for $x > 2$]

$$= 4 = f(3) \quad \textbf{(1)}$$

[$\because f(3) = 3 + 1 = 4$]

$\therefore f(x)$ is continuous at $x = 3$. **(1/2)**

18. Do same as Q. No. 12. **[Ans.** $a = 3$ and $b = -2$]

19. Let $f(x) = \begin{cases} k(x^2 + 2), & \text{if } x \le 0 \\ 3x + 1, & \text{if } x > 0 \end{cases}$

is continuous at $x = 0$. **(1/2)**

Then, $\text{LHL} = \text{RHL} = f(0)$ ...(i) **(1)**

Here, $\text{LHL} = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} k(x^2 + 2)$

$$= \lim_{h \to 0} k[(0 - h)^2 + 2]$$

[put $x = 0 - h$; when $x \to 0^-$, then $h \to 0$]

$$= \lim_{h \to 0} k(h^2 + 2)$$

$$\Rightarrow \quad \text{LHL} = 2k \quad \textbf{(1/2)}$$

and $\text{RHL} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (3x + 1)$

$= \lim_{h \to 0} [3(0+h)+1]$

[put $x = 0 + h$; when $x \to 0^+$, then $h \to 0$]

$= \lim_{h \to 0} (3h+1)$

$\Rightarrow$ RHL $= 1$ **(1/2)**

From Eq. (i), we have

LHL = RHL

$\Rightarrow \quad 2k = 1$

$\therefore \quad k = \frac{1}{2}$ **(1/2)**

Now, let us check the continuity of the given function $f(x)$ at $x = 1$.

Consider, $\lim_{x \to 1} f(x) = \lim_{x \to 1} 3x + 1$

[$\because f(x) = 3x + 1$ for $x > 0$]

$= 4 = f(1)$ [$\because f(1) = 3+1 = 4$]

$\therefore f(x)$ is continuous at $x = 1$ **(1)**

20. First, verify continuity of the given function at $x = -3$ and $x = 3$. Then, point at which the given function is discontinuous will be the point of discontinuity.

Given function is

$$f(x) = \begin{cases} |x| + 3, & x \le -3 \\ -2x, & -3 < x < 3 \\ 6x + 2, & x \ge 3 \end{cases} = \begin{cases} -x + 3, & x \le -3 \\ -2x, & -3 < x < 3 \\ 6x + 2, & x \ge 3 \end{cases}$$

First, we verify continuity at $x = -3$ and then at $x = 3$

Continuity at $x = -3$

LHL $= \lim_{x \to (-3)^-} f(x) = \lim_{x \to (-3)^-} (-x + 3)$

$\Rightarrow$ LHL $= \lim_{h \to 0} [-(-3-h) + 3]$

$= \lim_{h \to 0} (3 + h + 3)$

[put $x = -3 - h$; when $x \to -3^-$, then $h \to 0$]

$= 3 + 3 = 6$ **(1/2)**

and RHL $= \lim_{x \to (-3)^+} f(x) = \lim_{x \to (-3)^+} (-2x)$

$\Rightarrow$ RHL $= \lim_{h \to 0} [-2(-3+h)]$

[put $x = -3 + h$; when $x \to -3^+$, then $h \to 0$]

$= \lim_{h \to 0} (6 - 2h)$

$\Rightarrow$ RHL $= 6$ **(1/2)**

Also, $f(-3) =$ value of $f(x)$ at $x = -3$

$= -(-3) + 3$

$= 3 + 3 = 6$ **(1/2)**

$\because$ LHL = RHL $= f(-3)$

$\therefore f(x)$ is continuous at $x = -3$. So, $x = -3$ is the point of continuity. **(1/2)**

Continuity at $x = 3$

LHL $= \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} [-(2x)]$

$\Rightarrow$ LHL $= \lim_{h \to 0} [-2(3-h)]$

[put $x = 3 - h$; when $x \to 3^-$, then $h \to 0$]

$= \lim_{h \to 0} (-6 + 2h)$

$\Rightarrow$ LHL $= -6$ **(1/2)**

and RHL $= \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (6x + 2)$

$\Rightarrow$ RHL $= \lim_{h \to 0} [6(3+h) + 2]$

[put $x = 3 + h$; when $x \to 3^+$, then $h \to 0$]

$= \lim_{h \to 0} (18 + 6h + 2)$

$\Rightarrow$ RHL $= 20$ **(1/2)**

$\because$ LHL $\ne$ RHL

$\therefore f$ is discontinuous at $x = 3$.

Now, as $f(x)$ is a polynomial function for $x < -3$, $-3 < x < 3$ and $x > 3$, so it is continuous in these intervals.

Hence, only $x = 3$ is the point of discontinuity of $f(x)$. **(1)**

Solutions

1. Let $y = e^{\sqrt{3x}}$

Then, $\frac{dy}{dx} = \frac{d(e^{\sqrt{3x}})}{dx} = \frac{3 \cdot e^{\sqrt{3x}}}{2 \cdot \sqrt{3x}} = \frac{3e^{\sqrt{3x}}}{2\sqrt{3x}}$ **(1)**

2. Given, $y = \cos(\sqrt{3x})$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\{\cos(\sqrt{3x})\} = -\sin(\sqrt{3x}) \cdot \frac{d}{dx}(\sqrt{3x})$$

$$= -\sin(\sqrt{3x}) \cdot \frac{1}{2\sqrt{3x}} \cdot 3 = -\frac{3\sin\sqrt{3x}}{2\sqrt{3x}}$$ **(1)**

3. Given, $f(x) = x + 1 \Rightarrow f(f(x)) = f(x) + 1$

$\Rightarrow \quad fof(x) = x + 1 + 1 \Rightarrow fof(x) = x + 2$

Now, $\frac{d}{dx}(fof)(x) = \frac{d}{dx}(x + 2) = 1$ **(1)**

4. Given, $f(x) = x + 7,$

$g(x) = x - 7, x \in R$

Now, $(fog)(x) = f[g(x)] = f(x - 7) = (x - 7) + 7$

$(fog)(x) = x$

On differentiate w.r.t. x, we get

$$\frac{d}{dx}(fog)(x) = \frac{d}{dx}(x) \Rightarrow \frac{d}{dx}(fog)(x) = 1$$ **(1)**

5. We have, $y = x|x|$

When, $x < 0$, then $|x| = -x$

$\therefore \quad y = x(-x) = -x^2 \Rightarrow \frac{dy}{dx} = -2x$ **(1)**

6. Let $y = \tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$

$$= \tan^{-1}\left(\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right)$$

$\left[\because 1 + \cos A = 2\cos^2\frac{A}{2} \text{ and } \sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}\right]$

$$= \tan^{-1}\left(\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}\right) = \tan^{-1}\left(\cot\frac{x}{2}\right)$$

$$= \frac{\pi}{2} - \cot^{-1}\left(\cot\frac{x}{2}\right)$$

$\left[\because \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}, \forall x \in R\right]$

$= \frac{\pi}{2} - \frac{x}{2}$ $\quad [\because \cot^{-1}(\cot x) = x, \forall x \in (0, \pi)]$ **(1)**

Now, on differentiating both sides w.r.t. x, we get

$\frac{dy}{dx} = 0 - \frac{1}{2} = -\frac{1}{2}$ **(1)**

7. Let $y = \tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right) = \tan^{-1}\left(\frac{1-\tan x}{1+\tan x}\right)$

$$= \tan^{-1}\left(\frac{\tan\frac{\pi}{4} - \tan x}{1 + \tan\frac{\pi}{4}\tan x}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4} - x\right)\right)$$

$= \frac{\pi}{4} - x$ $\quad [\because \tan^{-1}(\tan\theta) = \theta]$ **(1)**

On differentiating w.r.t. x, we get

$\frac{dy}{dx} = -1$ **(1)**

8. Given, $f(x) = x^3 - 3x$ in $[-\sqrt{3}, 0]$ ...(i)

We know that, according to Rolle's theorem, if $f(x)$ is continuous in $[a, b]$ differentiable in (a, b) and $f(a) = f(b)$, then there exist $c \in (a, b)$ such that $f'(c) = 0$.

Here $f(x)$, being a polynomial function, is continuous in $[-\sqrt{3}, 0]$ and differentiable in $(-\sqrt{3}, 0)$.

Also, $f(-\sqrt{3}) = 0 = f(0)$

$\therefore \quad f'(c) = 0$, for some $c \in (-\sqrt{3}, 0)$ ...(ii) **(1)**

Now, $\quad f'(x) = 3x^2 - 3$ [from Eq. (i)]

$\Rightarrow \quad f'(c) = 3c^2 - 3 = 0$ [from Eq. (ii)]

$\Rightarrow \quad c = \pm 1$

But $c \in (-\sqrt{3}, 0)$, so neglecting positive value of c.

$\therefore \quad c = -1$ **(1)**

9. We have, $\quad \sin^2 y + \cos xy = K$

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}(\sin^2 y + \cos xy) = \frac{d}{dx}(K)$$

$\Rightarrow \quad \frac{d}{dx}(\sin^2 y) + \frac{d}{dx}(\cos xy) = 0$

$\Rightarrow \quad 2\sin y\cos y\frac{dy}{dx} + (-\sin xy)\frac{d}{dx}(xy) = 0$

$\Rightarrow \quad \sin 2y\frac{dy}{dx} - \sin xy\left(x\frac{dy}{dx} + y \cdot 1\right) = 0$

$\Rightarrow \quad \sin 2y\frac{dy}{dx} - x\sin xy\frac{dy}{dx} = y\sin xy$

$\Rightarrow \quad \frac{dy}{dx} = \frac{y\sin(xy)}{\sin 2y - x\sin(xy)}$ **(1)**

$\therefore \quad \left(\frac{dy}{dx}\right)$ at $x = 1, y = \frac{\pi}{4}$

$$= \frac{\frac{\pi}{4}\sin\left(1.\frac{\pi}{4}\right)}{\sin\left(2.\frac{\pi}{4}\right) - 1\sin\left(1.\frac{\pi}{4}\right)}$$

$$= \frac{\frac{\pi}{4}\sin\left(\frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{4}\right)} = \frac{\frac{\pi}{4}\left(\frac{1}{\sqrt{2}}\right)}{1 - \frac{1}{\sqrt{2}}}$$

$= \frac{\frac{\pi}{4\sqrt{2}}}{(\sqrt{2}-1)/\sqrt{2}} = \frac{\pi}{4\sqrt{2}} \times \frac{\sqrt{2}}{(\sqrt{2}-1)} = \frac{\pi}{4(\sqrt{2}-1)}$ **(1)**

10. Given, $y = \sin^{-1}(6x\sqrt{1-9x^2})$

$\Rightarrow \quad y = \sin^{-1}(2 \cdot 3x\sqrt{1-(3x)^2})$

Put $\quad 3x = \sin\theta$, then

$y = \sin^{-1}(2\sin\theta\sqrt{1-\sin^2\theta})$

$\Rightarrow \quad y = \sin^{-1}(2\sin\theta \cdot \cos\theta)$

$\Rightarrow \quad y = \sin^{-1}(\sin 2\theta) \Rightarrow y = 2\theta$

$\Rightarrow \quad y = 2\sin^{-1}(3x)$ $\quad [\because \theta = \sin^{-1}(3x)]$ **(1)**

$\Rightarrow \quad \frac{dy}{dx} = \frac{2}{\sqrt{1-9x^2}}(3) \Rightarrow \frac{dy}{dx} = \frac{6}{\sqrt{1-9x^2}}$ **(1)**

11. First, take log on both sides, then differentiate both sides by using product rule.

Given, $(\cos x)^y = (\cos y)^x$

On taking log both sides, we get

$$\log (\cos x)^y = \log (\cos y)^x$$

$$\Rightarrow \quad y \log (\cos x) = x \log(\cos y)$$

$[\because \log x^n = n \log x]$ **(1)**

On differentiating both sides w.r.t. x, we get

$$y \cdot \frac{d}{dx} \log (\cos x) + \log \cos x \cdot \frac{d}{dx}(y) = x \frac{d}{dx} \log (\cos y) + \log (\cos y) \frac{d}{dx}(x)$$

[by using product rule of derivative] **(1)**

$$\Rightarrow \quad y \cdot \frac{1}{\cos x} \frac{d}{dx}(\cos x) + \log (\cos x) \frac{dy}{dx} = x \cdot \frac{1}{\cos y} \frac{d}{dx}(\cos y) + \log \cos y \cdot 1$$

$$\Rightarrow y \cdot \frac{1}{\cos x}(-\sin x) + \log(\cos x) \cdot \frac{dy}{dx} = x \frac{1}{\cos y}(-\sin y) \cdot \frac{dy}{dx} + \log(\cos y) \cdot 1 \quad \textbf{(1)}$$

$$\Rightarrow -y \tan x + \log (\cos x) \frac{dy}{dx} = -x \tan y \frac{dy}{dx} + \log(\cos y)$$

$$\Rightarrow [x \tan y + \log (\cos x)] \frac{dy}{dx} = \log(\cos y) + y \tan x$$

$$\therefore \quad \frac{dy}{dx} = \frac{\log(\cos y) + y \tan x}{x \tan y + \log(\cos x)} \quad \textbf{(1)}$$

12. First, solve the given equation and convert it into $y = f(x)$ form. Then, differentiate to get the required result.

To prove $\frac{dy}{dx} = -\frac{1}{(1+x)^2}$

Given equation is $x\sqrt{1+y} + y\sqrt{1+x} = 0$, where $x \neq y$, we first convert the given equation into $y = f(x)$ form.

Clearly, $\quad x\sqrt{1+y} = -y\sqrt{1+x}$

On squaring both sides, we get

$$x^2 (1+y) = y^2 (1+x)$$

$$\Rightarrow \quad x^2 + x^2 y = y^2 + y^2 x$$

$$\Rightarrow \quad x^2 - y^2 = y^2 x - x^2 y$$

$$\Rightarrow \quad (x-y)(x+y) = -xy(x-y)$$

$[\because a^2 - b^2 = (a-b)(a+b)]$

$$\Rightarrow \quad (x-y)(x+y) + xy(x-y) = 0 \quad \textbf{(1)}$$

$$\Rightarrow \quad (x-y)(x+y+xy) = 0$$

$\therefore$ Either $\quad x - y = 0$ or $x + y + xy = 0$

Now, $\quad x - y = 0 \Rightarrow x = y$

But it is given that $x \neq y$.

So, it is a contradiction.

$\therefore \quad x - y = 0$ is rejected.

Now, consider $y + xy + x = 0$

$$\Rightarrow \quad y(1+x) = -x \Rightarrow y = \frac{-x}{1+x} \quad \ldots\text{(i)} \ \textbf{(1)}$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(1+x) \times \frac{d}{dx}(-x) - (-x) \times \frac{d}{dx}(1+x)}{(1+x)^2} \quad \textbf{(1)}$$

[by using quotient rule of derivative]

$$\Rightarrow \frac{dy}{dx} = \frac{(1+x)(-1) + x(1)}{(1+x)^2} \Rightarrow \frac{dy}{dx} = \frac{-1-x+x}{(1+x)^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{-1}{(1+x)^2} \quad \textbf{(1)}$$

Hence proved.

13. Given, $y = (\sin^{-1} x)^2$ ...(i)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 2\sin^{-1} x \cdot \frac{1}{\sqrt{1-x^2}} \quad \textbf{(1)}$$

Again differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{(\sqrt{1-x^2})\left(\frac{2}{\sqrt{1-x^2}}\right) - \left(\frac{\frac{1}{2}(-2x)}{\sqrt{1-x^2}}\right) \cdot (2\sin^{-1} x)}{(\sqrt{1-x^2})^2}$$

$$= \frac{2 + \frac{2x \sin^{-1} x}{\sqrt{1-x^2}}}{1-x^2} \quad \textbf{(1)}$$

Now consider, $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2$

$$= (1-x^2)\left\{\frac{2 + \frac{2x \sin^{-1} x}{\sqrt{1-x^2}}}{1-x^2}\right\} - x\left\{\frac{2\sin^{-1} x}{\sqrt{1-x^2}}\right\} - 2 \quad \textbf{(1)}$$

$$= 2 + \frac{2x \sin^{-1} x}{\sqrt{1-x^2}} - \frac{2x \sin^{-1} x}{\sqrt{1-x^2}} - 2$$

$$= 0$$

Hence proved. (1)

14. Given, $(x-a)^2+(y-b)^2=c^2$

Differentiating w.r.t. x, we get

$$\frac{d\{(x-a)^2+(y-b)^2\}}{dx}=\frac{d(c^2)}{dx}$$

$$\Rightarrow \frac{d\{(x-a)^2\}}{dx}+\frac{d\{(y-b)^2\}}{dx}=0$$

$$\Rightarrow 2(x-a)\cdot\frac{d(x-a)}{dx}+2(y-b)\cdot\frac{d(y-b)}{dx}=0$$

$$\Rightarrow 2(x-a)\cdot(1-0)+2(y-b)\cdot\left(\frac{dy}{dx}-0\right)=0$$

$$\Rightarrow 2(x-a)+2(y-b)\cdot\left(\frac{dy}{dx}\right)=0$$

$$\Rightarrow 2(y-b)\cdot\frac{dy}{dx}=-2(x-a)$$

$$\Rightarrow \frac{dy}{dx}=-\frac{(x-a)}{(y-b)} \quad (1)$$

Again differentiating w.r.t. x, we get

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left\{-\frac{(x-a)}{(y-b)}\right\}$$

$$\frac{d^2y}{dx^2}=-\frac{\left\{(y-b)\frac{d(x-a)}{dx}-(x-a)\frac{d(y-b)}{dx}\right\}}{(y-b)^2}$$

$$=-\frac{\left\{(y-b)\cdot(1-0)-(x-a)\left(\frac{dy}{dx}-0\right)\right\}}{(y-b)^2}$$

$$=-\frac{\left\{(y-b)-\frac{dy}{dx}(x-a)\right\}}{(y-b)^2} \quad (1)$$

$$=-\frac{\left\{(y-b)+\frac{(x-a)}{(y-b)}(x-a)\right\}}{(y-b)^2}$$

$$=-\frac{\{(y-b)^2+(x-a)^2\}}{(y-b)^3}$$

$$=-\frac{c^2}{(y-b)^3} \quad [\because (x-a)^2+(y-b)^2=c^2] \quad (1)$$

Now,
$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}=\frac{\left[1+\left\{-\frac{(x-a)}{(y-b)}\right\}^2\right]^{3/2}}{\frac{-c^2}{(y-b)^3}}$$

$$=-\frac{\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^2}\right]^{3/2}}{\frac{c^2}{(y-b)^3}}=-\frac{\left[\frac{c^2}{(y-b)^2}\right]^{3/2}}{\frac{c^2}{(y-b)^3}}$$

$$=-\left[\frac{c^2}{(y-b)^2}\right]^{3/2}\times\frac{(y-b)^3}{c^2}$$

$$=-\left(\frac{c}{y-b}\right)^{2\times\frac{3}{2}}\times\frac{(y-b)^3}{c^2}=-\frac{c^3}{c^2}\times\frac{(y-b)^3}{(y-b)^3}$$

$$=-c$$

which is constant independent of a and b.

Hence proved. (1)

15. Given, $x=ae^t(\sin t+\cos t)$

and $y=ae^t(\sin t-\cos t)$ (1)

$$\therefore \frac{dx}{dt}=a\,[e^t(\cos t-\sin t)+e^t(\sin t+\cos t)]$$

$$=-y+x=x-y \quad (1)$$

and
$$\frac{dy}{dt}=a\,[e^t(\sin t-\cos t)+e^t(\sin t+\cos t)]$$

$$=y+x=x+y \quad (1)$$

Now, $\dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}\Rightarrow\dfrac{dy}{dx}=\dfrac{x+y}{x-y}$

Hence proved. (1)

16. Let $y=x^{\sin x}+(\sin x)^{\cos x}$

$$\Rightarrow y=e^{\sin x\log x}+e^{\cos x\cdot\log\sin x}$$

$$[\because a^b=e^{\log a^b}=e^{b\log a}] \quad (1)$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx}=e^{\sin x\cdot\log x}\frac{d}{dx}(\sin x\cdot\log x)$$

$$+e^{\cos x\cdot\log\sin x}\frac{d}{dx}(\cos x\cdot\log\sin x) \quad (1)$$

$$=x^{\sin x}\left\{\log x\frac{d}{dx}(\sin x)+\sin x\frac{d}{dx}(\log x)\right\}$$

$$+(\sin x)^{\cos x}\left\{\log\sin x\frac{d}{dx}(\cos x)+\cos x\cdot\frac{d}{dx}(\log\sin x)\right\}$$

$$=x^{\sin x}\left\{\cos x\cdot\log x+\frac{\sin x}{x}\right\}+(\sin x)^{\cos x}$$

$$\left\{-\sin x\log(\sin x)+\cos x\times\frac{1}{\sin x}\times\cos x\right\} \quad (1)$$

$$=x^{\sin x}\left\{\cos x\cdot\log x+\frac{\sin x}{x}\right\}+(\sin x)^{\cos x}$$

$$\left\{-\sin x\log(\sin x)+\frac{\cos^2 x}{\sin x}\right\} \quad (1)$$

17. $\log(x^2+y^2) = 2\tan^{-1}\left(\frac{y}{x}\right)$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{x^2+y^2}\left(2x+2y\frac{dy}{dx}\right) = 2\cdot\frac{1}{1+\frac{y^2}{x^2}}\left(\frac{y'\cdot x-y}{x^2}\right)$$ **(1)**

$$\Rightarrow \frac{2(x+y\cdot y')}{x^2+y^2} = \frac{2x^2}{x^2+y^2}\left(\frac{y'x-y}{x^2}\right)$$

$$\left[\because y' = \frac{dy}{dx}\right]$$ **(1)**

$$\Rightarrow x+y\cdot y' = y'x-y$$

$$\Rightarrow y'(x-y) = x+y$$

$$\Rightarrow y' = \frac{x+y}{x-y}$$ **(1)**

$$\Rightarrow \frac{dy}{dx} = \frac{x+y}{x-y}$$

Hence proved. (1)

18. Given, $x^y - y^x = a^b$

Let $x^y = u$ and $y^x = v$

Then, $u - v = a^b \Rightarrow \frac{du}{dx} - \frac{dv}{dx} = 0$ ...(i) **(1)**

Now, $u = x^y \Rightarrow \log u = y\log x$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \frac{y}{x} + \log x\frac{dy}{dx}$$

$$\Rightarrow \frac{du}{dx} = y\cdot x^{y-1} + x^y\cdot\log x\frac{dy}{dx}$$ **(1)**

and $v = y^x \Rightarrow \log v = x\log y$

$$\Rightarrow \frac{1}{v}\cdot\frac{dv}{dx} = \frac{x}{y}\frac{dy}{dx} + \log y$$

$$\Rightarrow \frac{dv}{dx} = xy^{x-1}\frac{dy}{dx} + y^x\log y$$ **(1)**

Now, Eq. (i) becomes,

$$y\cdot x^{y-1} + x^y\cdot\log x\frac{dy}{dx} - xy^{x-1}\frac{dy}{dx} - y^x\log y = 0$$

$$\Rightarrow \frac{dy}{dx}(x^y\log x - xy^{x-1}) = y^x\log y - y\cdot x^{y-1}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^x\cdot\log y - y\cdot x^{y-1}}{x^y\cdot\log x - x\cdot y^{x-1}}$$ **(1)**

19. Given that, $y = \sin t$

$$\Rightarrow \frac{dy}{dt} = \cos t$$ [differentiate w.r.t. t] ...(i)

$$\Rightarrow \frac{d^2y}{dt^2} = -\sin t$$ [differentiate w.r.t. t]

$$\left[\frac{d^2y}{dt^2}\right]_{t=\frac{\pi}{4}} = -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$ **(1)**

Again, $x = \cos t + \log\tan\frac{t}{2}$

$$\Rightarrow \frac{dx}{dt} = -\sin t + \frac{1}{\tan\frac{t}{2}}\cdot\sec^2\frac{t}{2}\cdot\frac{1}{2}$$

[differentiate w.r.t. t] **(1)**

$$= -\sin t + \frac{\cos\frac{t}{2}}{2\cdot\sin\frac{t}{2}}\cdot\frac{1}{\cos^2\frac{t}{2}}$$

$$= -\sin t + \frac{1}{\sin 2\times\frac{t}{2}}$$

$[\because 2\sin a\cos a = \sin 2a]$

$$= -\sin t + \operatorname{cosec} t$$ ...(ii) **(1)**

Now, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t}{\operatorname{cosec} t - \sin t}$

[using Eqs.(i) and (ii)]

$$= \frac{\cos t}{1-\sin^2 t}\cdot\sin t = \frac{\sin t\cdot\cos t}{\cos^2 t}$$

$$\frac{dy}{dx} = \tan t$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\sec^2 t}{\operatorname{cosec} t - \sin t}$$

$$= \frac{\sec^2 t\cdot\sin t}{\cos^2 t} = \sec^3 t\cdot\tan t$$

$$\Rightarrow \left[\frac{d^2y}{dx^2}\right]_{t=\frac{\pi}{4}} = \sec^3\frac{\pi}{4}\cdot\tan\frac{\pi}{4} = 2\sqrt{2}\times 1 = 2\sqrt{2}$$ **(1)**

20. Given, $y = \sin(\sin x)$ ...(i)

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \cos(\sin x)\cdot\cos x$$ ...(ii) **(1/2)**

Again, on differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \cos(\sin x)\cdot(-\sin x)$$

$$+\cos x(-\sin(\sin x))\cdot\cos x$$ **(1)**

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{\cos x}\cdot\left(\frac{dy}{dx}\right)(-\sin x) - y\cos^2 x \quad \textbf{(1/2)}$$

[using Eqs. (i) and (ii)]

$$\Rightarrow \frac{d^2y}{dx^2} = -\tan x\frac{dy}{dx} - y\cos^2 x$$

$$\Rightarrow \frac{d^2y}{dx^2} + \tan x\frac{dy}{dx} + y\cos^2 x = 0$$

Hence proved. **(2)**

21. We have, $(x^2 + y^2)^2 = xy$

On differentiating both sides w.r.t. x, we get

$$2(x^2 + y^2)\left[2x + 2y\frac{dy}{dx}\right] = \left[x\frac{dy}{dx} + y\right] \quad \textbf{(1)}$$

$$\Rightarrow 4(x^2 + y^2)\left(x + y\frac{dy}{dx}\right) = \left(y + x\frac{dy}{dx}\right)$$

$$\Rightarrow 4(x^2 + y^2)x + 4(x^2 + y^2)y\frac{dy}{dx} = y + x\frac{dy}{dx} \quad \textbf{(1)}$$

$$\Rightarrow \frac{dy}{dx}[4(x^2 + y^2)y - x] = y - 4x(x^2 + y^2)$$

$$\Rightarrow \frac{dy}{dx} = \frac{y - 4x(x^2 + y^2)}{4(x^2 + y^2)y - x}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{[y - 4x(x^2 + y^2)]}{[x - 4y(x^2 + y^2)]} \quad \textbf{(2)}$$

22. We have, $x = a(2\theta - \sin 2\theta)$

and $y = a(1 - \cos 2\theta)$

Clearly, $$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} \quad \ldots(i)$$

(1)

Here, $\frac{dx}{d\theta} = a(2 - \cos 2\theta\cdot 2) = 2a(1 - \cos 2\theta)$

and $\frac{dy}{d\theta} = a(0 + 2\sin 2\theta) = 2a\sin 2\theta$ **(1)**

From Eq. (i), we get

$$\frac{dy}{dx} = \frac{2a\sin 2\theta}{2a(1 - \cos 2\theta)}$$

$$= \frac{\sin 2\theta}{1 - \cos 2\theta}$$

$$= \frac{2\sin\theta\cos\theta}{2\sin^2\theta}$$

$[\because \cos 2\theta = 1 - 2\sin^2\theta]$

$= \cot\theta$ **(1)**

Now, $\left(\frac{dy}{dx}\right)_{\theta=\pi/3} = \cot\frac{\pi}{3} = \frac{1}{\sqrt{3}}$ **(1)**

23. Given, $\sin y = x\cos(a + y)$ ...(i)

$$\Rightarrow x = \frac{\sin y}{\cos(a + y)}$$

On differentiating both sides w.r.t. y, we get

$$\frac{dx}{dy} = \frac{\cos(a + y)\frac{d}{dy}(\sin y) - \sin y\frac{d}{dy}\cos(a + y)}{\cos^2(a + y)} \quad \textbf{(1)}$$

[using quotient rule of derivative]

$$\frac{dx}{dy} = \frac{\cos(a + y)\cos y + \sin y\sin(a + y)}{\cos^2(a + y)}$$

$$= \frac{\cos(a + y - y)}{\cos^2(a + y)} \quad \textbf{(1)}$$

$[\because \cos A\cos B + \sin A\sin B = \cos(A - B)]$

$$\Rightarrow \frac{dx}{dy} = \frac{\cos a}{\cos^2(a + y)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos^2(a + y)}{\cos a} \quad \textbf{(1)}$$

Put $x = 0$ in Eq. (i), we get

$y = 0$

Now, $\frac{dy}{dx} = \frac{\cos^2(a + 0)}{\cos a} = \frac{\cos^2 a}{\cos a} = \cos a$

Hence proved. **(1)**

24. Given, $y = a\tan^3\theta$ and $x = a\sec^3\theta$

On differentiating w.r.t. θ, we get

$$\frac{dy}{d\theta} = 3a\tan^2\theta\frac{d}{d\theta}(\tan\theta)$$

and $$\frac{dx}{d\theta} = 3a\sec^2\theta\frac{d}{d\theta}(\sec\theta)$$

$$\Rightarrow \frac{dy}{d\theta} = 3a\tan^2\theta\sec^2\theta$$

and $\frac{dx}{d\theta} = 3a\sec^2\theta\sec\theta\tan\theta$ **(1)**

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a\tan^2\theta\sec^2\theta}{3a\sec^2\theta\sec\theta\tan\theta}$$

$$\frac{dy}{dx} = \frac{\tan\theta}{\sec\theta} = \sin\theta \quad \textbf{(1)}$$

Again differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}(\sin\theta) = \frac{d}{d\theta}(\sin\theta)\frac{d\theta}{dx}$$

$$= \frac{\cos\theta}{3a\sec^3\theta\tan\theta} = \frac{\cos^5\theta}{3a\sin\theta} \quad \textbf{(1)}$$

At $\theta = \frac{\pi}{3}$,

$$\frac{d^2y}{dx^2} = \frac{\cos^5\frac{\pi}{3}}{3a\sin\frac{\pi}{3}} = \frac{\left(\frac{1}{2}\right)^5}{3a\left(\frac{\sqrt{3}}{2}\right)}$$

$$= \frac{1\times 2}{2^5\times 3a\sqrt{3}} = \frac{1}{48\sqrt{3}a}$$ **(1)**

25. We have, $y = e^{\tan^{-1}x}$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = e^{\tan^{-1}x}\frac{d}{dx}(\tan^{-1}x)$$

$$\Rightarrow \frac{dy}{dx} = e^{\tan^{-1}x}\times\frac{1}{(1+x^2)}$$

$$\Rightarrow (1+x^2)\frac{dy}{dx} = e^{\tan^{-1}x}$$ ...(i) **(1)**

Again differentiating both sides w.r.t. x, we get

$$(1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = e^{\tan^{-1}x}\times\frac{1}{(1+x^2)}$$ **(1)**

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2} + (2x)\frac{dy}{dx} = \frac{dy}{dx}$$ [from Eq. (i)] **(1)**

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2} + (2x-1)\frac{dy}{dx} = 0$$

Hence proved. **(1)**

26. Do same as Q. No. 18.

$$\left[\text{Ans. } \frac{dy}{dx} = \frac{-x^{y-1}\cdot y - y^x\log y}{x^y\log x + y^{x-1}\cdot x}\right]$$

27. Given, $e^y(x+1) = 1$

On taking log both sides, we get

$\log[e^y(x+1)] = \log 1$

$\Rightarrow \log e^y + \log(x+1) = \log 1$

$\Rightarrow y + \log(x+1) = \log 1$ [$\because \log e^y = y$] **(1)**

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} + \frac{1}{x+1} = 0$$ ...(i) **(1)**

Again, differentiating both sides w.r.t. 'x', we get

$$\frac{d^2y}{dx^2} - \frac{1}{(x+1)^2} = 0$$ **(1)**

$$\Rightarrow \frac{d^2y}{dx^2} - \left(-\frac{dy}{dx}\right)^2 = 0$$ [from Eq. (i)]

$$\Rightarrow \frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$$ **Hence proved.** **(1)**

28. Given, $y = x^x$

On taking log both sides, we get

$$\log y = \log x^x$$

$$\Rightarrow \log y = x\log x$$ **(1)**

On differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = x\frac{d}{dx}(\log x) + \log x\frac{d}{dx}(x)$$

[by using product rule of derivative]

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = x\times\frac{1}{x} + \log x\cdot 1$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = (1+\log x)$$

$$\Rightarrow \frac{dy}{dx} = y(1+\log x)$$ ...(i) **(1)**

Again, differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = y\frac{d}{dx}(1+\log x) + (1+\log x)\frac{dy}{dx}$$ **(1)**

[by using product rule of derivative]

$$\Rightarrow \frac{d^2y}{dx^2} = y\times\frac{1}{x} + (1+\log x)\frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{y}{x} + (1+\log x)\frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{y}{x} + \frac{1}{y}\left(\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right)$$ [using Eq. (i)]

$$\therefore \frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0$$ **(1)**

Hence proved.

29. Let $u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

Put $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$, then

$$u = \tan^{-1}\left[\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right]$$

$$= \tan^{-1}\left[\frac{\sqrt{\sec^2\theta}-1}{\tan\theta}\right]$$

$$= \tan^{-1}\left[\frac{\sec\theta-1}{\tan\theta}\right]$$

$= \tan^{-1}\left[\frac{1-\cos\theta}{\sin\theta}\right]$

$= \tan^{-1}\left[\frac{2\sin^2\theta/2}{2\sin\theta/2\cdot\cos\theta/2}\right]$

$= \tan^{-1}[\tan\theta/2]$

$\Rightarrow \quad u = \frac{\theta}{2} = \frac{1}{2}\tan^{-1} x \qquad [\because \tan^{-1}(\tan\theta) = \theta]$ (1)

On differentiating both sides w.r.t. x, we get

$\frac{du}{dx} = \frac{1}{2(1+x^2)} \qquad \left[\because \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}\right]$ (1/2)

Also, let $v = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Put $x = \tan\theta \Rightarrow \theta = \tan^{-1} x$, then we get

$v = \sin^{-1}\left[\frac{2\tan\theta}{1+\tan^2\theta}\right]$

$\Rightarrow \quad v = \sin^{-1}[\sin 2\theta] \Rightarrow v = 2\theta$

$\Rightarrow \quad v = 2\tan^{-1} x$ (1)

On differentiating both sides w.r.t. x, we get

$\frac{dv}{dx} = \frac{2}{1+x^2}$ ...(ii) (1/2)

Now, $\frac{du}{dv} = \frac{du}{dx} \times \frac{dx}{dv} = \frac{1}{2(1+x^2)} \times \frac{(1+x^2)}{2}$

[from Eqs. (i) and (ii)]

$\therefore \quad \frac{du}{dv} = \frac{1}{4}$ (1)

30. Given, $x = a\sin 2t(1+\cos 2t)$

and $y = b\cos 2t(1-\cos 2t)$

On differentiating x and y separately w.r.t. t, we get

$\frac{dx}{dt} = a\left[\sin 2t\frac{d}{dt}(1+\cos 2t) + (1+\cos 2t)\frac{d}{dt}(\sin 2t)\right]$

[by using product rule of derivative]

$= a[\sin 2t \times (0 - 2\sin 2t) + (1+\cos 2t)(2\cos 2t)]$

$= a(-2\sin^2 2t + 2\cos 2t + 2\cos^2 2t)$

$= a[2(\cos^2 2t - \sin^2 2t) + 2\cos 2t]$

$= a(2\cos 4t + 2\cos 2t) = 2a(\cos 4t + \cos 2t)$

$[\because \cos^2 2\theta - \sin^2 2\theta = \cos 4\theta]$

$= 2a\left[2\cos\left(\frac{4t+2t}{2}\right)\cdot\cos\left(\frac{4t-2t}{2}\right)\right]$

$\left[\because \cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cdot\cos\left(\frac{x-y}{2}\right)\right]$

$= 4a\cos 3t\cos t$ (1)

and $\frac{dy}{dt}$

$= b\left[\cos 2t\frac{d}{dt}(1-\cos 2t) + (1-\cos 2t)\frac{d}{dt}(\cos 2t)\right]$

[by using product rule of derivative]

$= b[\cos 2t \times (0 + 2\sin 2t) + (1-\cos 2t)(-2\sin 2t)]$

$= b(2\sin 2t\cos 2t - 2\sin 2t + 2\sin 2t\cos 2t)$

$= 2b(2\sin 2t\cos 2t - \sin 2t)$

$= 2b(\sin 4t - \sin 2t) \qquad [\because 2\sin 2\theta\cos 2\theta = \sin 4\theta]$

$= 2b\left[2\cos\left(\frac{4t+2t}{2}\right)\sin\left(\frac{4t-2t}{2}\right)\right]$

$\left[\because \sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\cdot\sin\left(\frac{x-y}{2}\right)\right]$

$= 4b\cos 3t\sin t$ (1)

Now, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$= \frac{4b\cos 3t\sin t}{4a\cos 3t\cdot\cos t}$

$= \frac{b}{a}\tan t$ (1)

At $t = \frac{\pi}{4}$, $\frac{dy}{dx} = \frac{b}{a}\tan\frac{\pi}{4} = \frac{b}{a}$

At $t = \frac{\pi}{3}$, $\frac{dy}{dx} = \frac{b}{a}\tan\frac{\pi}{3} = \frac{\sqrt{3}b}{a}$ (1)

31. Given, $x\cos(a+y) = \cos y$

$\Rightarrow \quad x = \frac{\cos y}{\cos(a+y)}$

On differentiating both sides w.r.t. y, we get

$\frac{dx}{dy} = \frac{\cos(a+y)\frac{d}{dy}\cos y - \cos y\frac{d}{dy}\cos(a+y)}{\cos^2(a+y)}$

[by using quotient rule of derivative]

$= \frac{\cos(a+y)\times(-\sin y) + \cos y \times \sin(a+y)}{\cos^2(a+y)}$ (1)

$= \frac{\sin(a+y)\cos y - \cos(a+y)\sin y}{\cos^2(a+y)}$

$\Rightarrow \quad \frac{dx}{dy} = \frac{\sin(a+y-y)}{\cos^2(a+y)} = \frac{\sin a}{\cos^2(a+y)}$

$[\because \sin A\cos B - \cos A\sin B = \sin(A-B)]$

$\Rightarrow \quad \frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$ ...(i) (1)

Again, on differentiating both sides of Eq. (1) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{1}{\sin a}\frac{d}{dx}\cos^2(a+y)$$

$$= \frac{1}{\sin a}\times\frac{d}{dy}\cos^2(a+y)\times\frac{dy}{dx} \quad (1)$$

$$= \frac{1}{\sin a}\times 2\cos(a+y)[-\sin(a+y)]\times\frac{dy}{dx}$$

$$= -\frac{2\sin(a+y)\cos(a+y)}{\sin a}\times\frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{\sin 2(a+y)}{\sin a}\frac{dy}{dx} \quad [\because 2\sin\theta\cos\theta = \sin 2\theta]$$

$$\therefore \sin a\frac{d^2y}{dx^2} + \sin 2(a+y)\frac{dy}{dx} = 0 \quad (1)$$

Hence proved.

32. Given, $y = \sin^{-1}\left[\frac{6x - 4\sqrt{1-4x^2}}{5}\right]$

Put $x = \frac{1}{2}\sin\theta$,

$$\therefore y = \sin^{-1}\left[\frac{6\times\frac{\sin\theta}{2} - 4\sqrt{1-4\times\left(\frac{\sin\theta}{2}\right)^2}}{5}\right] \quad (1/2)$$

$$= \sin^{-1}\left(\frac{3\sin\theta - 4\sqrt{1-\sin^2\theta}}{5}\right)$$

$$= \sin^{-1}\left(\frac{3\sin\theta - 4\cos\theta}{5}\right)$$

$$= \sin^{-1}\left(\frac{3}{5}\sin\theta - \frac{4}{5}\cos\theta\right) \quad \text{... (i) (1)}$$

Let $\cos\phi = \frac{3}{5}$, then $\sin\phi = \sqrt{1-\cos^2\phi} = \sqrt{1-\left(\frac{3}{5}\right)^2}$

$$= \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5} \quad (1/2)$$

Now, Eq. (i) becomes

$$y = \sin^{-1}(\cos\phi\sin\theta - \sin\phi\cos\theta)$$

$$= \sin^{-1}[\sin(\theta-\phi)] = \theta - \phi$$

$$\Rightarrow y = \sin^{-1}(2x) - \cos^{-1}\left(\frac{3}{5}\right)$$

$$\left[\because x = \frac{1}{2}\sin\theta \Rightarrow \sin\theta = 2x \Rightarrow \theta = \sin^{-1}(2x)\right.$$

$$\left.\text{and } \cos\phi = \frac{3}{5} \Rightarrow \phi = \cos^{-1}\frac{3}{5}\right] \quad (1)$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(2x)^2}}\frac{d}{dx}(2x) - 0$$

$$= \frac{2}{\sqrt{1-4x^2}} \quad (1)$$

33. Given, $f(x) = \begin{cases} x^2+3x+a, & x \le 1 \\ bx+2, & x>1 \end{cases}$ is differentiable at $x = 1$.

$$\therefore \quad Lf'(1) = Rf'(1) \quad \text{...(i) (1/2)}$$

Here, $Lf'(1) = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}$

$$= \lim_{h\to 0}\frac{(1-h)^2+3(1-h)+a-(4+a)}{-h}$$

$$= \lim_{h\to 0}\frac{1+h^2-2h+3-3h+a-4-a}{-h}$$

$$= \lim_{h\to 0}\frac{h^2-5h}{-h}$$

$$= \lim_{h\to 0} 5-h = 5 \quad (1)$$

and $Rf'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$

$$= \lim_{h\to 0}\frac{b(1+h)+2-(4+a)}{h}$$

$$= \lim_{h\to 0}\frac{b+bh+2-4-a}{h}$$

$$= \lim_{h\to 0}\frac{bh+b-a-2}{h}$$

Clearly, for $Rf'(1)$ to be exist $b-a-2$ should be equal to 0, i.e.

$$b-a-2=0 \quad \text{...(ii) (1)}$$

Now, $Rf'(1) = \lim_{h\to 0}\frac{bh}{h}$

$$= \lim_{h\to 0} b = b \quad (1/2)$$

From Eq. (i), we have

$$Lf'(1) = Rf'(1)$$

$$\Rightarrow 5 = b$$

$$\Rightarrow b = 5$$

Now, on substituting $b = 5$ in Eq. (ii), we get

$$5-a-2=0$$

$$\Rightarrow a = 3$$

Hence, $a = 3$ and $b = 5$

(1)

34. Given, $x = \sin t$ and $y = \sin pt$

On differentiating x and y separately w.r.t. t, we get

$$\frac{dx}{dt} = \cos t \text{ and } \frac{dy}{dt} = \cos pt \cdot p \Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos pt \cdot p}{\cos t} \quad \textbf{(1)}$$

Now, on differentiating both sides w.r.t x, we get

$$\frac{d^2y}{dx^2} = p\frac{[\cos t(-\sin pt \cdot p) - \cos pt(-\sin t)]}{\cos^2 t} \cdot \frac{dt}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{p[\cos pt \cdot \sin t - \cos t \sin pt \cdot p]}{\cos^2 t} \cdot \frac{1}{\cos t} \quad \textbf{(1)}$$

$$\Rightarrow \cos^2 t \frac{d^2y}{dx^2} = \frac{p\cos pt \cdot \sin t - p^2 \cos t \sin pt}{\cos t}$$

$$\Rightarrow (1 - \sin^2 t)\frac{d^2y}{dx^2} = \frac{p\cos pt}{\cos t} \cdot \sin t - p^2 \sin pt \quad \textbf{(1)}$$

$$\Rightarrow (1 - x^2)\frac{d^2y}{dx^2} = \frac{dy}{dx} \cdot x - p^2 y$$

$$\left[\because \frac{dy}{dx} = \frac{p\cos pt}{\cos t}, x = \sin t \text{ and } y = \sin pt\right]$$

$$\therefore (1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2 y = 0 \quad \textbf{(1)}$$

Hence proved.

35. First, put $x^2 = \sin\theta$, then reduce it in simplest form. Further, differentiate it.

Given, $y = \tan^{-1}\left(\frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}}\right)$

Put $x^2 = \sin\theta \Rightarrow \theta = \sin^{-1} x^2$

$$\therefore y = \tan^{-1}\left(\frac{\sqrt{1+\sin\theta} + \sqrt{1-\sin\theta}}{\sqrt{1+\sin\theta} - \sqrt{1-\sin\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} + \sqrt{\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} - 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}}{\sqrt{\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} - \sqrt{\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} - 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}}\right)$$

$$= \tan^{-1}\left[\frac{\sqrt{\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)^2} + \sqrt{\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)^2}}{\sqrt{\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)^2} - \sqrt{\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)^2}}\right] \quad \textbf{(1)}$$

$$= \tan^{-1}\left[\frac{\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right) + \left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)}{\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right) - \left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)}\right]$$

$$= \tan^{-1}\left(\frac{2\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}}\right) = \tan^{-1}\left(\cot\frac{\theta}{2}\right) \quad \textbf{(1)}$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right] \quad \left[\because \cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)\right]$$

$$= \frac{\pi}{2} - \frac{\theta}{2} \quad [\because \tan^{-1}(\tan\theta) = \theta]$$

$$\Rightarrow y = \frac{\pi}{2} - \frac{1}{2}\sin^{-1} x^2 \quad [\text{put } \theta = \sin^{-1} x^2] \quad \textbf{(1)}$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = -\frac{1}{2}\frac{1}{\sqrt{1-(x^2)^2}}(2x)$$

$$= \frac{-x}{\sqrt{1-x^4}} \quad \textbf{(1)}$$

36. Given, $x = a\cos\theta + b\sin\theta$ ...(i)

and $y = a\sin\theta - b\cos\theta$ ...(ii)

On differentiating both sides of Eqs. (i) and (ii) w.r.t. θ, we get

$$\frac{dx}{d\theta} = -a\sin\theta + b\cos\theta \quad \text{...(iii)} \ \textbf{(1)}$$

and $$\frac{dy}{d\theta} = a\cos\theta + b\sin\theta \quad \text{...(iv)} \ \textbf{(1)}$$

Now, $$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\cos\theta + b\sin\theta}{b\cos\theta - a\sin\theta}$$

[dividing Eq. (iv) by Eq. (iii)]

$$\Rightarrow \frac{dy}{dx} = \frac{x}{-y} \quad \text{[from Eqs. (i) and (ii)] ...(v)} \ \textbf{(1)}$$

Again, differentiating both sides of Eq. (v) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -\left[\frac{y \cdot 1 - x \cdot \frac{dy}{dx}}{y^2}\right]$$

[by using quotient rule of derivative]

$$\Rightarrow y^2\frac{d^2y}{dx^2} = -\left[y - x\frac{dy}{dx}\right]$$

$$\therefore y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0 \quad \textbf{(1)}$$

Hence proved.

37. Given, $f(x) = |x+1| + |x-1|, \forall x \in R$

It can be rewritten as

$$f(x) = \begin{cases} -(x+1)-(x-1), & x < -1 \\ (x+1)-(x-1), & -1 \le x < 1 \\ (x+1)+(x-1), & x \ge 1 \end{cases}$$

$$= \begin{cases} -2x, & x < -1 \\ 2, & -1 \le x < 1 \\ 2x, & x \ge 1 \end{cases} \quad \textbf{(1)}$$

Differentiability at $x = -1$

$$L f'(-1) = \lim_{h \to 0} \frac{f(-1-h) - f(-1)}{-h}$$

$$= \lim_{h \to 0} \frac{-2(-1-h) - 2}{-h}$$

$$= \lim_{h \to 0} \frac{2 + 2h - 2}{-h} = \lim_{h \to 0} (-2) = -2 \quad \textbf{(1/2)}$$

$$R f'(-1) = \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h}$$

$$= \lim_{h \to 0} \frac{2-2}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

$\because L f'(-1) \neq R f'(-1)$

$\therefore$ f is not differentiable at $x = -1$. **(1/2)**

Differentiability at $x = 1$

$$L f'(1) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h \to 0} \frac{2-2}{-h} = \lim_{h \to 0} \frac{0}{-h} = 0 \quad \textbf{(1/2)}$$

and $R f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{2(1+h) - 2}{h}$

$$= \lim_{h \to 0} \frac{2 + 2h - 2}{h} = \lim_{h \to 0} \frac{2h}{h} = 2 \quad \textbf{(1/2)}$$

$\because \quad L f'(1) \neq R f'(1)$

$\therefore$ f is not differentiable at $x = 1$.

Hence, f is not differentiable at $x = 1$ and -1. **(1)**

38. Given, $\quad y = e^{m \sin^{-1} x}$ ...(i)

On differentiating both sides of Eq.(i) w.r.t. x, we get

$$\frac{dy}{dx} = e^{m \sin^{-1} x} \frac{d}{dx}(m \sin^{-1} x)$$

[by using chain rule of derivative]

$$= e^{m \sin^{-1} x} \cdot m \frac{1}{\sqrt{1-x^2}} \Rightarrow \sqrt{1-x^2} \frac{dy}{dx} = my$$

[from Eq. (i)] **(1)**

Now, on squaring both sides, we get

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 = m^2 y^2 \quad \text{...(ii)}$$

On differentiating both sides of Eq. (ii) w.r.t. x, we get

$$(1-x^2)\, 2\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2(-2x) = 2m^2 y\left(\frac{dy}{dx}\right) \quad \textbf{(1)}$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = m^2 y \quad \textbf{(1)}$$

$\left[\text{dividing both sides by } 2\left(\frac{dy}{dx}\right)\right]$

$$\therefore \quad (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2 y = 0 \quad \textbf{(1)}$$

Hence proved.

39. Given, $f(x) = \sqrt{x^2+1}$, $g(x) = \frac{x+1}{x^2+1}$

and $\quad h(x) = 2x - 3$

On differentiating above functions w.r.t. x, we get

$$f'(x) = \frac{1}{2\sqrt{1+x^2}} \times 2x = \frac{x}{\sqrt{1+x^2}} \quad \textbf{(1)}$$

[by using chain rule of derivative]

$$g'(x) = \frac{(x^2+1) \cdot 1 - (x+1)2x}{(x^2+1)^2}$$

[by using quotient rule of derivative]

$$= \frac{x^2 + 1 - 2x^2 - 2x}{(x^2+1)^2} = \frac{-x^2 - 2x + 1}{(x^2+1)^2}$$

and $h'(x) = 2$ **(1)**

Now, $f'[h'\{g'(x)\}] = f'\left[h'\left(\frac{-x^2 - 2x + 1}{(x^2+1)^2}\right)\right]$

$$= f'(2) \quad [\because h'(x) = 2] \quad \textbf{(1)}$$

$$= \frac{2}{\sqrt{1+4}} = \frac{2}{\sqrt{5}} \quad \left[\because f'(x) = \frac{x}{\sqrt{1+x^2}}\right] \quad \textbf{(1)}$$

40. Given, $y = \left(x + \sqrt{1+x^2}\right)^n$ ...(i)

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = n(x + \sqrt{1+x^2})^{n-1}\left(1 + \frac{2x}{2\sqrt{1+x^2}}\right)$$

[by using chain rule of derivative] **(1)**

$$\Rightarrow \quad \frac{dy}{dx} = n(x + \sqrt{1+x^2})^{n-1}\left(\frac{x + \sqrt{1+x^2}}{\sqrt{1+x^2}}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{n(x+\sqrt{1+x^2})^n}{\sqrt{1+x^2}}$$ **(1)**

$$\Rightarrow \frac{dy}{dx} = \frac{ny}{\sqrt{1+x^2}}$$ [from Eq. (i)]

$$\Rightarrow \sqrt{1+x^2}\frac{dy}{dx} = ny$$ ...(ii) **(1)**

Again, differentiating both sides w.r.t. x, we get

$$\sqrt{1+x^2}\frac{d^2y}{dx^2} + \frac{2x}{2\sqrt{1+x^2}}\cdot\frac{dy}{dx} = n\frac{dy}{dx}$$

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = n\cdot\sqrt{1+x^2}\frac{dy}{dx}$$

[multiplying both sides by $\sqrt{1+x^2}$]

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = n\sqrt{1+x^2}\cdot\frac{ny}{\sqrt{1+x^2}}$$

[from Eq. (ii)]

$$\therefore (1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = n^2y$$ **(1)**

Hence proved.

41. Given, $f(x) = \begin{cases} x, & x<1 \\ 2-x, & 1\le x\le 2 \\ -2+3x-x^2, & x>2 \end{cases}$

Differentiability at $x=1$

$$\text{LHD} = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}$$

$$= \lim_{h\to 0}\frac{(1-h)-[2-(1)]}{-h} = \lim_{h\to 0}\frac{-h}{-h} = 1$$ **(1/2)**

$$\text{RHD} = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0}\frac{2-(1+h)-(2-1)}{h}$$

$$= \lim_{h\to 0}\frac{-h}{h} = -1$$ **(1/2)**

$\because$ LHD $\neq$ RHD

So, $f(x)$ is not differentiable at $x=1$. **(1/2)**

Differentiability at $x=2$

$$\text{LHD} = \lim_{h\to 0}\frac{f(2-h)-f(2)}{-h}$$

$$= \lim_{h\to 0}\frac{2-(2-h)-(2-2)}{-h}$$

$$= \lim_{h\to 0}\frac{h}{-h} = -1$$ **(1/2)**

$$\text{RHD} = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h\to 0}\frac{-2+3(2+h)-(2+h)^2-(2-2)}{h}$$

$$= \lim_{h\to 0}\frac{-2+6+3h-(4+h^2+4h)-0}{h}$$

$$= \lim_{h\to 0}\frac{-h^2-h}{h}$$

$$= \lim_{h\to 0}\frac{-h(h+1)}{h} = -(0+1) = -1$$ **(1/2)**

$\because$ LHD = RHD

So, $f(x)$ is differentiable at $x=2$ **(1/2)**

Hence, $f(x)$ is not differentiable at $x=1$, but it differentiable at $x=2$ **(1)**

42. Given function is

$$f(x) = \begin{cases} \lambda(x^2+2), & \text{if } x\le 0 \\ 4x+6, & \text{if } x>0 \end{cases}$$

Let $f(x)$ is continuous at $x=0$.

Then, LHL = RHL = $f(0)$ ...(i)

Here, $\text{RHL} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(4x+6)$

$$= \lim_{h\to 0}[4(0+h)+6]$$ **(1/2)**

[put $x = 0+h$; when $x\to 0^+$, then $h\to 0$]

$$= \lim_{h\to 0}(4h+6)$$

$$= 4\times 0+6 = 6$$

and $f(0) = \lambda(0^2+2) = 2\lambda$.

From Eq. (i), RHL = $f(0)$

$\Rightarrow$ $2\lambda = 6 \Rightarrow \lambda = 3$ **(1)**

Now, given function becomes

$$f(x) = \begin{cases} 3(x^2+2), & \text{if } x\le 0 \\ 4x+6, & \text{if } x>0 \end{cases}$$ **(1)**

Now, let us check the differentiability at $x=0$.

$$\text{LHD} = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}$$

$$= \lim_{h\to 0}\frac{3[(0-h)^2+2]-3(0+2)}{-h}$$

$$= \lim_{h\to 0}\frac{3[h^2+2]-6}{-h} = \lim_{h\to 0}(-3h) = 0$$ **(1/2)**

$$\text{and RHD} = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0}\frac{[4(0+h)+6]-3(0+2)}{h} = \lim_{h\to 0}\frac{4h}{h} = 4$$ **(1/2)**

$\because$ LHD $\neq$ RHD

$\therefore$ $f(x)$ is not differentiable at $x=0$. **(1/2)**

43. Given, $y = (\sin x)^x + \sin^{-1}\sqrt{x}$ ...(i)

Let $u = (\sin x)^x$ ...(ii)

Then, Eq. (i) becomes, $y = u + \sin^{-1}\sqrt{x}$ ...(iii)

On taking log both sides of Eq. (ii), we get

$$\log u = x \log \sin x \quad (1)$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{u}\frac{du}{dx} = x\frac{d}{dx}(\log \sin x) + \log \sin x \frac{d}{dx}(x)$$

[by using product rule of derivative]

$$\Rightarrow \frac{du}{dx} = u\left[x \times \frac{1}{\sin x}\frac{d}{dx}(\sin x) + \log \sin x (1)\right] \quad (1)$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^x \left[\frac{x}{\sin x} \times \cos x + \log \sin x\right]$$

[from Eq. (ii)]

$$\Rightarrow \frac{du}{dx} = (\sin x)^x [x \cot x + \log \sin x] \quad ...(iv)$$

On differentiating both sides of Eq.(iii) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{1}{\sqrt{1-(\sqrt{x})^2}}\frac{d}{dx}(\sqrt{x}) \quad (1)$$

$$\therefore \frac{dy}{dx} = (\sin x)^x [x \cot x + \log \sin x] + \frac{1}{\sqrt{1-x}} \times \frac{1}{2\sqrt{x}}$$ [from Eq. (iv)] (1)

44. We have, $y = \dfrac{x\cos^{-1} x}{\sqrt{1-x^2}} - \log\sqrt{1-x^2}$ ...(i)

On differentiating both sides of Eq. (i) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x\cos^{-1}x}{\sqrt{1-x^2}}\right) - \frac{d}{dx}(\log\sqrt{1-x^2}) \quad (1/2)$$

$$= \frac{\sqrt{1-x^2}\left[x \cdot \frac{(-1)}{\sqrt{1-x^2}} + \cos^{-1} x\right] - x\cos^{-1}x \cdot \frac{1}{2\sqrt{1-x^2}}(-2x)}{(\sqrt{1-x^2})^2} - \frac{1}{\sqrt{1-x^2}} \cdot \frac{(-2x)}{2\sqrt{1-x^2}} \quad (1)$$

$$= \frac{-x + \sqrt{1-x^2}\cos^{-1}x + \frac{x^2\cos^{-1}x}{\sqrt{1-x^2}}}{(\sqrt{1-x^2})^2} + \frac{x}{(\sqrt{1-x^2})^2} \quad (1)$$

$$= \frac{-x + \sqrt{1-x^2}\cos^{-1}x + \frac{x^2\cos^{-1}x}{\sqrt{1-x^2}} + x}{(\sqrt{1-x^2})^2} \quad (1/2)$$

$$= \frac{(1-x^2)\cos^{-1}x + x^2\cos^{-1}x}{(\sqrt{1-x^2})^3} = \frac{\cos^{-1}x}{(1-x^2)^{3/2}} \quad (1)$$

Hence proved.

45. Let $u = \sin x$

On differentiating both sides w.r.t. x, we get

$$\frac{du}{dx} = \cos x \quad ...(i)\ (1)$$

Also, let $v = \cos x$

On differentiating both sides w.r.t. x, we get

$$\frac{dv}{dx} = -\sin x \quad ...(ii)\ (1)$$

Now, $\dfrac{du}{dv} = \dfrac{du}{dx} \times \dfrac{dx}{dv} = -\dfrac{\cos x}{\sin x}$ (1)

[from Eqs. (i) and (ii)]

$$\therefore \frac{du}{dv} = -\cot x \quad (1)$$

46.

> First, convert the given expression in $\sin^{-1}[x\sqrt{1-y^2} - y\sqrt{1-x^2}]$ form and then put $x = \sin\phi$ and $y = \sin\theta$. Now, simplify the resulting expression and differentiate it.

Given, $y = \sin^{-1}[x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}]$

Above equation can be rewritten as

$$y = \sin^{-1}[x\sqrt{1-(\sqrt{x})^2} - \sqrt{x}\sqrt{1-x^2}] \quad (1/2)$$

Now, put $\sqrt{x} = \sin\theta$ and $x = \sin\phi$, so that

$$y = \sin^{-1}[\sin\phi\sqrt{1-\sin^2\theta} - \sin\theta\sqrt{1-\sin^2\phi}] \quad (1/2)$$

$$\Rightarrow y = \sin^{-1}[\sin\phi\cos\theta - \sin\theta\cos\phi]$$

$[\because \sqrt{1-\sin^2 x} = \cos x]$

$$\Rightarrow y = \sin^{-1}\sin(\phi - \theta) \quad (1)$$

$[\because \sin\phi\cos\theta - \cos\phi\sin\theta = \sin(\phi-\theta)]$

$$\Rightarrow y = \phi - \theta \quad [\because \sin^{-1}\sin\theta = \theta]$$

$$\Rightarrow y = \sin^{-1}x - \sin^{-1}\sqrt{x}$$

$[\because \phi = \sin^{-1}x \text{ and } \theta = \sin^{-1}\sqrt{x}]$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-(\sqrt{x})^2}} \times \frac{d}{dx}(\sqrt{x}) \quad (1)$$

$\left[\because \dfrac{d}{d\theta}(\sin^{-1}\theta) = \dfrac{1}{\sqrt{1-\theta^2}}\right]$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}$$

Hence, $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}} - \dfrac{1}{2\sqrt{x-x^2}}$ (1)

47. Given, $e^x + e^y = e^{x+y}$ ...(i)

On dividing Eq. (i) by e^{x+y}, we get

$$e^{-y} + e^{-x} = 1 \quad \text{...(ii) (1)}$$

On differentiating both sides of Eq. (ii) w.r.t. x, we get

$$e^{-y}\cdot\left(\frac{-dy}{dx}\right) + e^{-x}(-1) = 0 \quad (1)$$

$$\Rightarrow \quad -e^{-y}\frac{dy}{dx} - e^{-x} = 0$$

$$\Rightarrow \quad -e^{-y}\frac{dy}{dx} = e^{-x}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{e^{-x}}{e^{-y}}$$

$$\Rightarrow \quad \frac{dy}{dx} = -e^{(y-x)}$$

$$\therefore \quad \frac{dy}{dx} + e^{(y-x)} = 0 \quad (2)$$

Hence proved.

48. Given, $x = ae^{\theta}(\sin\theta - \cos\theta)$

On differentiating both sides w.r.t. θ, we get

$$\frac{dx}{d\theta} = a\frac{d}{d\theta}[e^{\theta}\sin\theta - e^{\theta}\cos\theta]$$

$$= a\left[\frac{d}{d\theta}(e^{\theta}\sin\theta) - \frac{d}{d\theta}(e^{\theta}\cos\theta)\right]$$

$$= a\left[e^{\theta}\frac{d}{d\theta}(\sin\theta) + \sin\theta\frac{d}{d\theta}(e^{\theta}) - e^{\theta}\frac{d}{d\theta}(\cos\theta) - \cos\theta\frac{d}{d\theta}(e^{\theta})\right] \quad (1)$$

[by using product rule of derivative]

$$= a[e^{\theta}\cos\theta + e^{\theta}\sin\theta - e^{\theta}(-\sin\theta) - e^{\theta}\cos\theta]$$

$$= a[e^{\theta}\cos\theta + e^{\theta}\sin\theta + e^{\theta}\sin\theta - e^{\theta}\cos\theta]$$

$$\Rightarrow \quad \frac{dx}{d\theta} = a[2e^{\theta}\sin\theta] = 2ae^{\theta}\sin\theta \quad \text{...(i) (1)}$$

Also, we have $y = ae^{\theta}(\sin\theta + \cos\theta)$

On differentiating both sides w.r.t. θ, we get

$$\frac{dy}{d\theta} = a\left[\frac{d}{d\theta}(e^{\theta}\sin\theta) + \frac{d}{d\theta}(e^{\theta}\cos\theta)\right]$$

$$= a\left[e^{\theta}\frac{d}{d\theta}(\sin\theta) + \sin\theta\frac{d}{d\theta}(e^{\theta}) + e^{\theta}\frac{d}{d\theta}(\cos\theta) + \cos\theta\frac{d}{d\theta}(e^{\theta})\right] \quad (1)$$

[by using product rule of derivative]

$$= a[e^{\theta}\cos\theta + e^{\theta}\sin\theta - e^{\theta}\sin\theta + e^{\theta}\cos\theta]$$

$$= a[2e^{\theta}\cos\theta]$$

$$\Rightarrow \quad \frac{dy}{d\theta} = 2ae^{\theta}\cos\theta \quad \text{...(ii)}$$

$$\text{Now, } \frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{2ae^{\theta}\cos\theta}{2ae^{\theta}\sin\theta}$$

[from Eqs. (i) and (ii)]

$$= \cot\theta$$

At $\theta = \frac{\pi}{4}$, $\frac{dy}{dx} = \cot\frac{\pi}{4} = 1$ $\left[\because \cot\frac{\pi}{4} = 1\right]$ (1)

49. Do same as Q. 19. $\left[\textbf{Ans. } \frac{8\sqrt{3}}{a}\right]$

50. First, take log on both sides. Further, differentiate it to prove the required result.

Given, $x^m y^n = (x+y)^{m+n}$

On taking log both sides, we get

$$\log(x^m y^n) = \log(x+y)^{m+n}$$

$$\Rightarrow \quad \log(x^m) + \log(y^n) = (m+n)\log(x+y)$$

$$\Rightarrow \quad m\log x + n\log y = (m+n)\log(x+y) \quad (1)$$

On differentiating both sides w.r.t. x, we get

$$\frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = \frac{m+n}{x+y}\left(1 + \frac{dy}{dx}\right) \quad (1)$$

$$\Rightarrow \quad \frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = \frac{m+n}{x+y} + \frac{m+n}{x+y}\frac{dy}{dx}$$

$$\Rightarrow \frac{m}{x} - \frac{(m+n)}{x+y} = \left(\frac{m+n}{x+y} - \frac{n}{y}\right)\frac{dy}{dx}$$

$$\Rightarrow \left[\frac{my + ny - nx - ny}{y(x+y)}\right]\frac{dy}{dx} = \frac{mx + my - mx - nx}{x(x+y)}$$

$$\Rightarrow \quad \frac{dy}{dx}\left[\frac{my - nx}{y}\right] = \frac{my - nx}{x}$$

$$\text{Hence,} \quad \frac{dy}{dx} = \frac{y}{x} \quad (2)$$

51. Let $u = \tan^{-1}\left[\frac{\sqrt{1-x^2}}{x}\right]$

Put $x = \cos\theta \Rightarrow \theta = \cos^{-1}x$

Then, $u = \tan^{-1}\left[\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}\right] = \tan^{-1}\left[\frac{\sqrt{\sin^2\theta}}{\cos\theta}\right]$

$[\because \cos^2\theta + \sin^2\theta = 1 \Rightarrow \sin^2\theta = 1 - \cos^2\theta]$

$$= \tan^{-1}\left[\frac{\sin\theta}{\cos\theta}\right] = \tan^{-1}[\tan\theta] = \theta$$

$\Rightarrow \quad u = \cos^{-1}x$ $\quad [\because x = \cos\theta]$ (1/2)

On differentiating both sides w.r.t. x, we get

$$\frac{du}{dx} = -\frac{1}{\sqrt{1-x^2}} \quad (1)$$

Again, let $v = \cos^{-1}(2x\sqrt{1-x^2})$

Put $x = \cos\theta \Rightarrow \theta = \cos^{-1} x$

Then, $v = \cos^{-1}[2\cos\theta\sqrt{1-\cos^2\theta}]$

$$= \cos^{-1}[2\cos\theta\sin\theta] \quad \left[\begin{array}{l}\because \sin\theta = \sqrt{1-\cos^2\theta} \\ \Rightarrow \sin^2\theta = 1-\cos^2\theta\end{array}\right]$$

$$= \cos^{-1}[\sin 2\theta] \quad (1/2)$$

$$= \cos^{-1}\left[\cos\left(\frac{\pi}{2} - 2\theta\right)\right] = \frac{\pi}{2} - 2\theta$$

$$\Rightarrow v = \frac{\pi}{2} - 2\cos^{-1} x \quad [\because \theta = \cos^{-1} x] \quad (1)$$

On differentiating both sides w.r.t. x, we get

$$\frac{dv}{dx} = \frac{2}{\sqrt{1-x^2}}$$

Now, $\frac{du}{dv} = \frac{du}{dx} \times \frac{dx}{dv} = -\frac{1}{\sqrt{1-x^2}} \times \frac{\sqrt{1-x^2}}{2}$

$$= -\frac{1}{2} \quad (1)$$

52. Let $u = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

Put $x = \sin\theta \Rightarrow \theta = \sin^{-1} x$, then

$$u = \tan^{-1}\left[\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\right]$$

$$\Rightarrow u = \tan^{-1}\left[\frac{\sin\theta}{\cos\theta}\right] \left[\begin{array}{l}\because \sin^2\theta + \cos^2\theta = 1 \\ \Rightarrow \cos\theta = \sqrt{(1-\sin^2\theta)}\end{array}\right]$$

$$\Rightarrow u = \tan^{-1}(\tan\theta) \Rightarrow u = \theta$$

$$\Rightarrow u = \sin^{-1} x \quad (1)$$

On differentiating both sides w.r.t. x, we get

$$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}} \quad \text{...(i)(1)}$$

Again, let $v = \sin^{-1}(2x\sqrt{1-x^2})$

Put $x = \sin\theta \Rightarrow \theta = \sin^{-1} x$, then

$$v = \sin^{-1}(2\sin\theta\sqrt{1-\sin^2\theta})$$

$$\Rightarrow v = \sin^{-1}(2\sin\theta\cos\theta)$$

$$\Rightarrow v = \sin^{-1}(\sin 2\theta) \Rightarrow v = 2\theta$$

$$\Rightarrow v = 2\sin^{-1} x \quad (1)$$

On differentiating both sides w.r.t. x, we get

$$\frac{dv}{dx} = \frac{2}{\sqrt{1-x^2}} \quad \text{...(ii)}$$

Now, $\frac{du}{dv} = \frac{du}{dx} \times \frac{dx}{dv}$

$$\Rightarrow \frac{du}{dv} = \frac{1}{\sqrt{1-x^2}} \times \frac{\sqrt{1-x^2}}{2}$$

$$\therefore \frac{du}{dv} = \frac{1}{2} \quad (1)$$

53. Given, $y = Pe^{ax} + Qe^{bx}$ ...(i)

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = P\frac{d}{dx}(e^{ax}) + Q\frac{d}{dx}(e^{bx})$$

$$\Rightarrow \frac{dy}{dx} = Pa\,e^{ax} + Qb\,e^{bx} \quad \text{...(ii)} \quad (1\tfrac{1}{2})$$

Again, differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = Pa\frac{d}{dx}(e^{ax}) + Qb\frac{d}{dx}(e^{bx})$$

$$= Pa(a\,e^{ax}) + Qb(b\,e^{bx})$$

$$= a^2P\,e^{ax} + b^2Q\,e^{bx} \quad \text{...(iii)} \quad (1\tfrac{1}{2})$$

Now, LHS $= \frac{d^2y}{dx^2} - (a+b)\frac{dy}{dx} + aby$

On putting the values from Eqs. (i), (ii) and (iii), we get

LHS $= a^2P\,e^{ax} + b^2Q\,e^{bx}$

$$-(a+b)(aP\,e^{ax} + bQ\,e^{bx}) + ab(P\,e^{ax} + Q\,e^{bx})$$

$$= a^2P\,e^{ax} + b^2Q\,e^{bx} - a^2P\,e^{ax} - ab\,Q\,e^{bx}$$

$$- ab\,P\,e^{ax} - b^2Q\,e^{bx} + ab\,P\,e^{ax} + ab\,Q\,e^{bx}$$

$$= 0 = \text{RHS} \quad (1)$$

Hence proved.

54. Given, $x = \cos t(3 - 2\cos^2 t)$

$$\Rightarrow x = 3\cos t - 2\cos^3 t$$

On differentiating both sides w.r.t. t, we get

$$\frac{dx}{dt} = 3(-\sin t) - 2(3)\cos^2 t(-\sin t)$$

$$\Rightarrow \frac{dx}{dt} = -3\sin t + 6\cos^2 t\sin t \quad \text{...(i) (1)}$$

Also, $y = \sin t(3 - 2\sin^2 t)$

$$\Rightarrow y = 3\sin t - 2\sin^3 t$$

On differentiating both sides w.r.t. t, we get

$$\frac{dy}{dt} = 3\cos t - 2 \times 3 \times \sin^2 t\cos t$$

$$\Rightarrow \quad \frac{dy}{dt} = 3\cos t - 6\sin^2 t \cos t \quad \text{...(ii) (1)}$$

Now, $$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{3\cos t - 6\cos t \sin^2 t}{-3\sin t + 6\cos^2 t \sin t}$$

[from Eqs. (i) and (ii)]

$$= \frac{\cos t - 2\cos t \sin^2 t}{-\sin t + 2\cos^2 t \sin t} \quad \text{(1)}$$

$$= \frac{\cos t(1 - 2\sin^2 t)}{\sin t(2\cos^2 t - 1)} = \frac{\cos t \cdot \cos 2t}{\sin t \cdot \cos 2t}$$

$$= \cot(t) \quad \text{(1/2)}$$

$$\therefore \quad \left.\frac{dy}{dx}\right|_{\text{at } t = \frac{\pi}{4}} = \cot\left(\frac{\pi}{4}\right) = 1 \quad \text{(1/2)}$$

55. Given, $(x - y) \cdot e^{\frac{x}{x-y}} = a$

On taking log both sides, we get

$$\log\left[(x - y) \cdot e^{\frac{x}{x-y}}\right] = \log a$$

$$\Rightarrow \quad \log(x - y) + \log e^{\frac{x}{x-y}} = \log a$$

$[\because \log(mn) = \log m + \log n]$

$$\Rightarrow \quad \log(x - y) + \frac{x}{x - y} \log_e e = \log a$$

$$\Rightarrow \quad \log(x - y) + \frac{x}{x - y} = \log a \quad \text{(1)}$$

$[\because \log_e e = 1]$

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}[\log(x - y)] + \frac{d}{dx}\left(\frac{x}{x - y}\right) = \frac{d}{dx}(\log a)$$

$$\Rightarrow \quad \frac{1}{x - y}\frac{d}{dx}(x - y) + \frac{(x - y)\frac{d}{dx}(x) - x\frac{d}{dx}(x - y)}{(x - y)^2} = 0$$

[by using quotient rule of derivative] (1)

$$\Rightarrow \quad \frac{1}{x - y} \cdot (1 - y') + \frac{(x - y) - x(1 - y')}{(x - y)^2} = 0 \quad \text{(1)}$$

where, $y' = dy/dx$

$$\Rightarrow \quad (x - y)(1 - y') + x - y - x(1 - y') = 0$$

$$\Rightarrow \quad yy' + x - 2y = 0$$

$$\therefore \quad y\frac{dy}{dx} + x = 2y \quad \text{(1)}$$

Hence proved.

56. Do same as Q. No. 19. $\left[\textbf{Ans. } \frac{8\sqrt{2}}{a\pi}\right]$

57. Given, $y = \tan^{-1}\left(\frac{a}{x}\right) + \log\sqrt{\frac{x - a}{x + a}}$

$$= \tan^{-1}\left(\frac{a}{x}\right) + \log\left(\frac{x - a}{x + a}\right)^{1/2}$$

$$\Rightarrow \quad y = \tan^{-1}\left(\frac{a}{x}\right) + \frac{1}{2}[\log(x - a) - \log(x + a)]$$

$$\left[\because \log\frac{m}{n} = \log m - \log n\right] \quad \text{(1)}$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{1 + \frac{a^2}{x^2}} \cdot \left(\frac{-a}{x^2}\right) + \frac{1}{2}\left[\frac{1}{x - a} - \frac{1}{x + a}\right] \quad \text{(1)}$$

$$\left[\because \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2} \text{ and } \frac{d}{dx}(\log x) = \frac{1}{x}\right]$$

$$= \frac{-a}{x^2 + a^2} + \frac{1}{2}\left[\frac{x + a - x + a}{(x - a)(x + a)}\right]$$

$$= \frac{-a}{x^2 + a^2} + \frac{a}{x^2 - a^2} = \frac{-x^2 a + a^3 + x^2 a + a^3}{(x^2 + a^2)(x^2 - a^2)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{2a^3}{x^4 - a^4} \quad [\because (a + b)(a - b) = a^2 - b^2] \quad \text{(2)}$$

Hence proved.

58. Let $u = (\tan^{-1} x)^y$ and $v = y^{\cot x}$

Then, given equation becomes $u + v = 1$

On differentiating both sides w.r.t. x, we get

$$\frac{du}{dx} + \frac{dv}{dx} = 0 \quad \text{...(i)}$$

Now, $\quad u = (\tan^{-1} x)^y$

On taking log both sides, we get

$$\log u = y \log(\tan^{-1} x)$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(y) \cdot \log(\tan^{-1} x) + y\frac{d}{dx}(\log \tan^{-1} x)$$

[by using product rule of derivative]

$$\Rightarrow \quad \frac{1}{u}\frac{du}{dx} = \frac{dy}{dx}\log(\tan^{-1} x) + \frac{y}{(\tan^{-1} x)(1 + x^2)}$$

$$\Rightarrow \quad \frac{du}{dx} = (\tan^{-1} x)^y\left[\frac{dy}{dx}\log(\tan^{-1} x) + \frac{y}{(\tan^{-1} x)(1 + x^2)}\right] \quad \text{...(ii) (1½)}$$

Also, $\quad v = y^{\cot x}$

On taking log both sides, we get

$$\log v = \cot x \log y$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{v}\frac{dv}{dx} = \frac{d}{dx}(\cot x)\cdot \log y + \cot x \frac{d}{dx}(\log y)$$

[by using product rule of derivative]

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = -\operatorname{cosec}^2 x \log y + \frac{\cot x}{y}\frac{dy}{dx}$$

$$\Rightarrow \frac{dv}{dx} = y^{\cot x}\left[-\operatorname{cosec}^2 x \log y + \frac{\cot x}{y}\frac{dy}{dx}\right]$$

...(iii) (1½)

On putting values from Eqs. (ii) and (iii) in Eq. (i), we get

$$(\tan^{-1} x)^y\left[\frac{dy}{dx}\log(\tan^{-1} x) + \frac{y}{(\tan^{-1} x)(1+x^2)}\right] + y^{\cot x}\left[-\operatorname{cosec}^2 x \log y + \frac{\cot x}{y}\frac{dy}{dx}\right] = 0$$

$$\Rightarrow \frac{dy}{dx}[(\tan^{-1} x)^y \log(\tan^{-1} x) + \cot x \cdot y^{\cot x - 1}]$$

$$= -\left[\frac{y}{1+x^2}(\tan^{-1} x)^{y-1} - y^{\cot x}\operatorname{cosec}^2 x \log y\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\left[\frac{y}{1+x^2}(\tan^{-1} x)^{y-1} - y^{\cot x}\cdot \operatorname{cosec}^2 x \log y\right]}{[(\tan^{-1} x)^y \log(\tan^{-1} x) + \cot x \cdot y^{\cot x - 1}]}$$ (1)

59. Given, $x = 2\cos\theta - \cos 2\theta$

and $y = 2\sin\theta - \sin 2\theta$

On differentiating both sides w.r.t. θ, we get

$$\frac{dx}{d\theta} = -2\sin\theta + 2\sin 2\theta$$

and $$\frac{dy}{d\theta} = 2\cos\theta - 2\cos 2\theta$$ (1)

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2(\cos\theta - \cos 2\theta)}{2(-\sin\theta + \sin 2\theta)}$$ (1)

$$= \frac{2\sin\left(\frac{\theta+2\theta}{2}\right)\sin\left(\frac{2\theta-\theta}{2}\right)}{2\left[\cos\left(\frac{2\theta+\theta}{2}\right)\sin\left(\frac{2\theta-\theta}{2}\right)\right]}$$ (1)

$$\left[\begin{array}{l}\because \cos C - \cos D = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right) \\ \text{and } \sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\end{array}\right]$$

$$= \frac{\sin\left(\frac{3\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{3\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)} = \tan\left(\frac{3\theta}{2}\right)$$ (1)

Hence proved.

60. Given, $y = x\log\left(\frac{x}{a+bx}\right)$ ...(i)

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = x\frac{d}{dx}\log\left(\frac{x}{a+bx}\right) + \log\left(\frac{x}{a+bx}\right)\frac{d}{dx}(x)$$

[by using product rule of derivative]

$$= x\left(\frac{1}{\frac{x}{a+bx}}\right)\frac{d}{dx}\left(\frac{x}{a+bx}\right) + \log\left(\frac{x}{a+bx}\right)\cdot 1$$

$$\left[\because \frac{d}{dx}(\log x) = \frac{1}{x}\right]$$

$$= (a+bx)\left[\frac{(a+bx)(1) - x(b)}{(a+bx)^2}\right] + \log\left(\frac{x}{a+bx}\right)$$

[by using quotient rule of derivative]

$$= (a+bx)\left[\frac{a}{(a+bx)^2}\right] + \log\left(\frac{x}{a+bx}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{a}{a+bx} + \log\left(\frac{x}{a+bx}\right)$$ (1)

$$\Rightarrow \frac{dy}{dx} = \frac{a}{a+bx} + \frac{y}{x}$$ [using Eq. (i)]

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = \frac{a}{a+bx}$$

$$\Rightarrow x\frac{dy}{dx} - y = \frac{ax}{a+bx}$$ ...(ii) (1)

On differentiating both sides w.r.t. x, we get

$$\Rightarrow x\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot(1) - \frac{dy}{dx} = a\left[\frac{(a+bx)\cdot(1) - x(b)}{(a+bx)^2}\right]$$

[by using quotient rule of derivative]

$$\Rightarrow x\frac{d^2y}{dx^2} = \frac{a^2}{(a+bx)^2}$$ (1)

$$\Rightarrow x^3\frac{d^2y}{dx^2} = \frac{a^2x^2}{(a+bx)^2} = \left[\frac{ax}{a+bx}\right]^2$$

[multiplying both sides by x^2]

$$\Rightarrow x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2$$ [using Eq. (ii)] (1)

Hence proved.

61. Given, $x = \cos\theta$ ...(i)

and $y = \sin^3\theta$ ...(ii)

On differentiating both sides of Eqs. (i) and (ii) w.r.t θ, we get

$$\frac{dx}{d\theta}=-\sin\theta \text{ and } \frac{dy}{d\theta}=3\sin^2\theta\cdot\cos\theta \quad \textbf{(1/2)}$$

$$\Rightarrow \frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{3\sin^2\theta\cos\theta}{-\sin\theta}$$

$$=-3\sin\theta\cos\theta=\frac{-3}{2}(2\sin\theta\cos\theta)$$

$$\Rightarrow \frac{dy}{dx}=\frac{-3}{2}\sin 2\theta \quad \text{...(iii)} \textbf{ (1)}$$

On differentiating both sides w.r.t x, we get

$$\frac{d^2y}{dx^2}=\frac{-3}{2}\cdot 2\cos 2\theta\cdot\frac{d\theta}{dx}=-3\cos 2\theta\cdot\left(\frac{-1}{\sin\theta}\right)$$

$$\left[\because \frac{d\theta}{dx}=-\frac{1}{\sin\theta}\right]$$

$$=\frac{3\cos 2\theta}{\sin\theta} \quad \text{...(iv)} \textbf{ (1)}$$

Now, consider LHS $= y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2$

$$=\sin^3\theta\left(\frac{3\cos 2\theta}{\sin\theta}\right)+\left(\frac{-3}{2}\sin 2\theta\right)^2 \quad \textbf{(1/2)}$$

$$=\sin^2\theta\cdot 3\cdot\cos 2\theta+\frac{9}{4}\sin^2 2\theta$$

$$=3\sin^2\theta(2\cos^2\theta-1)+\frac{9}{4}(4\sin^2\theta\cos^2\theta)$$

$$=6\sin^2\theta\cos^2\theta-3\sin^2\theta+9\sin^2\theta\cos^2\theta$$

$$=15\sin^2\theta\cos^2\theta-3\sin^2\theta$$

$$=3\sin^2\theta(5\cos^2\theta-1)$$ **Hence proved. (1)**

62. Let $y=(\log x)^x+x^{\log x}$

Also, let $u=(\log x)^x$ and $v=x^{\log x}$, then $y=u+v$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \quad \text{...(i)} \textbf{ (1)}$$

Now, consider $u=(\log x)^x$

On taking log both sides, we get

$$\log u=\log(\log x)^x=x\log(\log x)$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{u}\frac{du}{dx}=x\cdot\frac{d}{dx}\log(\log x)+\log(\log x)\cdot\frac{d}{dx}(x)$$

$$=\frac{x}{\log x}\cdot\frac{1}{x}+\log(\log x)$$

$$\frac{du}{dx}=u\left[\frac{1}{\log x}+\log(\log x)\right]$$

$$\Rightarrow \frac{du}{dx}=(\log x)^x\left[\frac{1}{\log x}+\log(\log x)\right] \quad \text{...(ii)} \textbf{ (1)}$$

$$[\because u=(\log x)^x]$$

Since, $v=x^{\log x}$

On taking log both sides, we get

$$\log v=\log(x^{\log x})=(\log x)(\log x)=(\log x)^2$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{v}\frac{dv}{dx}=2\log x\cdot\frac{1}{x}\Rightarrow \frac{dv}{dx}=v\left[\frac{2\log x}{x}\right]$$

$$\Rightarrow \frac{dv}{dx}=x^{\log x}\left[\frac{2\log x}{x}\right] \quad [\because v=x^{\log x}] \text{...(iii)} \textbf{ (1)}$$

From Eqs. (i), (ii) and (iii), we get

$$\frac{dy}{dx}=(\log x)^x\left\{\frac{1}{\log x}+\log(\log x)\right\}+2\left(\frac{\log x}{x}\right)x^{\log x} \quad \textbf{(1)}$$

63. Given, $y=\log[x+\sqrt{x^2+a^2}]$

On differentiating both sides w.r.t. x, we get

$$\therefore \frac{dy}{dx}=\frac{1}{x+\sqrt{x^2+a^2}}\frac{d}{dx}\left(x+\sqrt{x^2+a^2}\right)$$

$$\left[\because \frac{d}{dx}(\log f(x))=\frac{1}{f(x)}\frac{d}{dx}f(x)\right] \textbf{ (1)}$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{x+\sqrt{x^2+a^2}}\left(1+\frac{2x}{2\sqrt{x^2+a^2}}\right)$$

$$\left[\because \frac{d}{dx}(\sqrt{x^2+a^2})=\frac{1}{2\sqrt{x^2+a^2}}\times 2x\right]$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{x+\sqrt{x^2+a^2}}\left(\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}}\right)$$

$$\Rightarrow \frac{dy}{dx}(\sqrt{x^2+a^2})=1 \quad \textbf{(1)}$$

Again, on differentiating both sides w.r.t. x, we get

$$\sqrt{x^2+a^2}\frac{d}{dx}\left(\frac{dy}{dx}\right)+\frac{dy}{dx}\cdot\frac{d}{dx}(\sqrt{x^2+a^2})=\frac{d(1)}{dx}$$

[by using product rule of derivative]

$$\Rightarrow \frac{d^2y}{dx^2}(\sqrt{x^2+a^2})+\frac{dy}{dx}\frac{1\cdot 2x}{2\sqrt{x^2+a^2}}=0 \quad \textbf{(1)}$$

On multiplying both sides by $\sqrt{x^2+a^2}$, we get

$$\frac{d^2y}{dx^2}(\sqrt{x^2+a^2})^2+\frac{dy}{dx}\times\frac{x\sqrt{x^2+a^2}}{\sqrt{x^2+a^2}}=0$$

$$\therefore (x^2+a^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=0$$ **Hence proved. (1)**

64. Given, $f(x)=|x-3|$

First, we check the continuity of $f(x)$ at $x=3$.

Here, $\text{LHL}=\lim_{x\to 3^-}|x-3|=\lim_{h\to 0}|3-h-3|$

[put $x=3-h$; when $x\to 3^-$, then $h\to 0$]

$=\lim_{h\to 0}|-h|=0$ **(1)**

$\text{RHL}=\lim_{x\to 3^+}|x-3|=\lim_{h\to 0}|3+h-3|=\lim_{h\to 0}|h|=0$

[put $x=3+h$; when $x\to 3^+$, then $h\to 0$]

and $f(3)=|3-3|=0$

Thus, $\text{LHL}=\text{RHL}=f(3)$

Hence, f is continuous at $x=3$. **(1)**

Now, let us check the differentiability of $f(x)$ at $x=3$.

$\text{LHD}=f'(3^-)=\lim_{h\to 0}\frac{f(3-h)-f(3)}{-h}$

$\left[\because Lf'(a)=\lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}\right]$

$=\lim_{h\to 0}\frac{|3-h-3|-|3-3|}{-h}$

$=\lim_{h\to 0}\frac{|-h|}{-h}=\lim_{h\to 0}\frac{h}{-h}=-1$

$[\because |-x|=x, \text{ if } x>0]$ **(1)**

$\text{RHD}=f'(3^+)=\lim_{h\to 0}\frac{f(3+h)-f(3)}{h}$

$\left[\because Rf'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\right]$

$=\lim_{h\to 0}\frac{|3+h-3|-|3-3|}{h}$

$=\lim_{h\to 0}\frac{|h|}{h}=\lim_{h\to 0}\frac{h}{h}=1$

$[\because |x|=x, \text{ if } x>0]$ **(1)**

Since, LHD $\neq$ RHD at $x=3$.

So, f is not differentiable. **Hence proved.**

65. Do same as Q. No. 19. $\left[\textbf{Ans. } \frac{d^2y}{dx^2}=\frac{-\operatorname{cosec}^2 t}{a\cos t}\right]$

66.

First, put 6^x equal to $\tan\theta$, so that it becomes some standard trigonometric function. Then, simplify the expression and then differentiate by using chain rule.

Let $y=\sin^{-1}\left[\frac{2^{x+1}\cdot 3^x}{1+(36)^x}\right]$

$=\sin^{-1}\left[\frac{2\cdot 2^x\cdot 3^x}{1+(6^2)^x}\right]=\sin^{-1}\left[\frac{2\cdot 6^x}{1+(6^x)^2}\right]$ **(1)**

Put $6^x=\tan\theta \Rightarrow \theta=\tan^{-1}(6^x)$

Then, $y=\sin^{-1}\left(\frac{2\cdot\tan\theta}{1+\tan^2\theta}\right)$

$=\sin^{-1}(\sin 2\theta)$ $\left[\because \sin 2\theta=\frac{2\tan\theta}{1+\tan^2\theta}\right]$ **(1)**

$=2\theta$

$\Rightarrow y=2\tan^{-1}(6^x)$ $[\because \theta=\tan^{-1}(6^x)]$

On differentiating both sides w.r.t. x, we get

$\frac{dy}{dx}=\frac{2}{1+(6^x)^2}\frac{d}{dx}(6^x)$ $\left[\because \frac{d}{dx}\tan^{-1}x=\frac{1}{1+x^2}\right]$ **(1)**

$\therefore \frac{dy}{dx}=\frac{2}{1+(6^x)^2}\cdot 6^x\cdot\log 6=\left[\frac{2^{x+1}\cdot 3^x}{1+(36)^x}\right]\log 6$ **(1)**

67. Given, $x=a\cos^3\theta$ and $y=a\sin^3\theta$

On differentiating both sides of x and y w.r.t. θ, we get

$\frac{dx}{d\theta}=3a\cos^2\theta\frac{d}{d\theta}(\cos\theta)=3a\cos^2\theta\cdot(-\sin\theta)$

$=-3a\cos^2\theta\cdot\sin\theta$ **(1)**

and $\frac{dy}{d\theta}=3a\sin^2\theta\frac{d}{d\theta}(\sin\theta)$

$=3a\sin^2\theta\cdot(\cos\theta)=3a\sin^2\theta\cdot\cos\theta$

Now, $\frac{dy}{dx}=\left(\frac{dy/d\theta}{dx/d\theta}\right)$

$=\frac{3a\sin^2\theta\cdot\cos\theta}{-3a\cos^2\theta\cdot\sin\theta}=-\tan\theta$ **(1)**

Again, on differentiating both sides w.r.t. x, we get

$\frac{d^2y}{dx^2}=\frac{d}{dx}(-\tan\theta)=-\frac{d}{d\theta}(\tan\theta)\frac{d\theta}{dx}$

$=-\sec^2\theta\cdot\frac{d\theta}{dx}$

$=-\sec^2\theta\cdot\left(\frac{-1}{3a\cos^2\theta\cdot\sin\theta}\right)$

$\left[\because \frac{d\theta}{dx}=\frac{-1}{3a\cos^2\theta\sin\theta}\right]$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{1}{3a\cos^4\theta \cdot \sin\theta} \quad (1)$$

$\therefore$ At $\theta = \frac{\pi}{6}$,

$$\left(\frac{d^2y}{dx^2}\right)_{\text{at } \theta = \frac{\pi}{6}} = \frac{1}{3a\left(\cos\frac{\pi}{6}\right)^4\left(\sin\frac{\pi}{6}\right)}$$

$$= \frac{1}{3a\left(\frac{\sqrt{3}}{2}\right)^4\left(\frac{1}{2}\right)}$$

$$= \frac{1}{3a\left(\frac{9}{16}\right)\left(\frac{1}{2}\right)} = \frac{32}{27a} \quad (1)$$

68. Given, $x\sin(a+y) + \sin a\cos(a+y) = 0$

$$\Rightarrow \quad x = \frac{-\sin a\cos(a+y)}{\sin(a+y)} \quad (1)$$

On differentiating both sides w.r.t. y, we get

$$\frac{dx}{dy} = \frac{-\left[\begin{array}{l}\sin(a+y)\frac{d}{dy}\{\sin a\cos(a+y)\} \\ -\sin a\cos(a+y)\frac{d}{dy}\{\sin(a+y)\}\end{array}\right]}{\sin^2(a+y)}$$

[by using quotient rule of derivative]

$$= \left\{\frac{\begin{array}{l}\sin(a+y) \cdot \sin a\sin(a+y) \\ +\sin a\cos(a+y)\cos(a+y)\end{array}}{\sin^2(a+y)}\right\} \quad (1)$$

$$= \frac{\sin a}{\sin^2(a+y)}\{\sin^2(a+y) + \cos^2(a+y)\}$$

$$= \frac{\sin a}{\sin^2(a+y)} \cdot 1 \quad [\because \sin^2\theta + \cos^2\theta = 1] \quad (1)$$

$$\therefore \quad \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$ **Hence proved.** (1)

69. First, take log on both sides and convert it into $y = f(x)$ form. Then, differentiate both sides to get required result.

Given, $\quad x^y = e^{x-y}$

On taking log both sides, we get

$$y\log_e x = (x-y)\log_e e$$

$$\Rightarrow \quad y\log_e x = x - y \quad [\because \log_e e = 1]$$

$$\Rightarrow \quad y(1+\log x) = x \quad \Rightarrow \quad y = \frac{x}{1+\log x} \quad (1)$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(1+\log x)\frac{d}{dx}(x) - x\frac{d}{dx}(1+\log x)}{(1+\log x)^2}$$

[by using quotient rule of derivative] (1)

$$= \frac{1+\log x - x \cdot \frac{1}{x}}{(1+\log x)^2} = \frac{1+\log x - 1}{(1+\log x)^2}$$

Hence, $$\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2} \quad (1)$$

Also, it can be written as

$$\frac{dy}{dx} = \frac{\log x}{(\log_e e + \log x)^2} \quad [\log_e e = 1]$$

$$\therefore \quad \frac{dy}{dx} = \frac{\log x}{\{\log(ex)\}^2}$$ **Hence proved.** (1)

70. Do same as Q. No. 69.

71. Do same as Q. No. 31.

72. Do same as Q. No. 13.

73. Given, $x = \sqrt{a^{\sin^{-1} t}}$ and $y = \sqrt{a^{\cos^{-1} t}}$

Consider, $x = (a^{\sin^{-1} t})^{1/2}$

On differentiating both sides w.r.t. t, we get

$$\frac{dx}{dt} = \frac{1}{2}(a^{\sin^{-1} t})^{-1/2}\frac{d}{dt}(a^{\sin^{-1} t}) \quad (1/2)$$

[by using chain rule of derivative]

$$= \frac{1}{2}(a^{\sin^{-1} t})^{-1/2} a^{\sin^{-1} t}\log a\frac{d}{dt}(\sin^{-1} t)$$

$$\left[\because \frac{d}{dx}(a^x) = a^x \log a\right]$$

$$= \frac{1}{2}(a^{\sin^{-1} t})^{-1/2} a^{\sin^{-1} t}\log a \cdot \frac{1}{\sqrt{1-t^2}}$$

$$= \frac{1}{2}(a^{\sin^{-1} t})^{1/2}\log a \cdot \frac{1}{\sqrt{1-t^2}}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{\frac{1}{2}\sqrt{a^{\sin^{-1} t}} \cdot \log a}{\sqrt{1-t^2}} \quad \ldots\text{(i)} \quad (1)$$

Now, consider $y = (a^{\cos^{-1} t})^{1/2}$

On differentiating both sides w.r.t. t, we get

$$\frac{dy}{dt} = \frac{1}{2}(a^{\cos^{-1} t})^{-1/2}\frac{d}{dt}(a^{\cos^{-1} t}) \quad (1/2)$$

[by using chain rule of derivative]

$$= \frac{1}{2}(a^{\cos^{-1}t})^{-1/2} a^{\cos^{-1}t} \log a \frac{d}{dt}(\cos^{-1} t)$$

$$\left[\because \frac{d}{dx}(a^x) = a^x \log a\right]$$

$$= \frac{1}{2}(a^{\cos^{-1}t})^{1/2} \log a \cdot \frac{(-1)}{\sqrt{1-t^2}}$$

$$\Rightarrow \frac{dy}{dt} = \frac{-\frac{1}{2}\sqrt{a^{\cos^{-1}t}} \cdot \log a}{\sqrt{1-t^2}} \quad \text{...(ii)} \quad \textbf{(1)}$$

On dividing Eq. (ii) by Eq. (i), we get

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\left(\frac{-\frac{1}{2}\sqrt{a^{\cos^{-1}t}}\log a}{\sqrt{1-t^2}}\right)}{\left(\frac{\frac{1}{2}\sqrt{a^{\sin^{-1}t}}\log a}{\sqrt{1-t^2}}\right)}$$

$$= -\frac{\sqrt{a^{\cos^{-1}t}}}{\sqrt{a^{\sin^{-1}t}}} = -\frac{y}{x} \quad \textbf{(1)}$$

$$\left[\text{given, } \sqrt{a^{\cos^{-1}t}} = y \text{ and } \sqrt{a^{\sin^{-1}t}} = x\right]$$

Hence proved.

74. Do same as Q. No. 29. $\left[\textbf{Ans. } \frac{1}{2(1+x^2)}\right]$

75. Given, $y = (\tan^{-1} x)^2$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 2\tan^{-1} x . \frac{1}{1+x^2} \quad \left[\because \frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{2\tan^{-1} x}{1+x^2} \Rightarrow (1+x^2)\frac{dy}{dx} = 2\tan^{-1} x \quad \textbf{(1½)}$$

Again, differentiating both sides w.r.t. x, we get

$$(1+x^2)\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\cdot\frac{d}{dx}(1+x^2) = \frac{d}{dx}(2\tan^{-1} x) \quad \textbf{(1)}$$

[by using product rule of derivative]

$$\Rightarrow (1+x^2).\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot 2x = \frac{2}{1+x^2}$$

$$\left[\because \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}\right]$$

On multiplying both sides by $(1+x^2)$, we get

$$(1+x^2)^2\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot 2x\cdot(1+x^2) = \frac{2}{1+x^2}\cdot(1+x^2)$$

$$\Rightarrow (1+x^2)^2\frac{d^2y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} = 2 \quad \textbf{(1½)}$$

Hence proved.

76. Given, $y = x^{\sin x - \cos x} + \frac{x^2-1}{x^2+1}$

Let $u = x^{\sin x - \cos x}$ and $v = \frac{x^2-1}{x^2+1}$

Then, the given equation becomes

$$y = u + v$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \text{...(i)}$$

[differentiate w.r.t. x] **(1/2)**

Consider $u = x^{\sin x - \cos x}$

On taking log both sides, we get

$$\log u = (\sin x - \cos x)\cdot \log x$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{u}\frac{du}{dx} = (\sin x - \cos x)\cdot\frac{1}{x} + \log x\cdot(\cos x + \sin x)$$

$$\Rightarrow \frac{du}{dx} = x^{\sin x - \cos x}\left[\frac{\sin x - \cos x}{x} + \log x\cdot(\cos x + \sin x)\right] \quad \text{...(ii)} \quad \textbf{(1)}$$

Now, consider $v = \frac{x^2-1}{x^2+1} = 1 - \frac{2}{x^2+1}$

On differentiating both sides w.r.t. x, we get

$$\frac{dv}{dx} = 0 - \frac{(x^2+1)\frac{d}{dx}(2) - 2\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

[by using quotient rule of derivative]

$$\Rightarrow \frac{dv}{dx} = -\left[\frac{0 - 2\cdot 2x}{(x^2+1)^2}\right] = \frac{4x}{(x^2+1)^2} \quad \text{...(iii)} \quad \textbf{(1½)}$$

On substituting the values from Eqs. (ii) and (iii) to Eq. (i), we get

$$\frac{dy}{dx} = x^{\sin x - \cos x}\left[\frac{\sin x - \cos x}{x} + \log x(\cos x + \sin x)\right] + \frac{4x}{(x^2+1)^2} \quad \textbf{(1)}$$

77. Given, $x = a(\cos t + t\sin t)$

On differentiating both sides w.r.t. t, we get

$$\frac{dx}{dt} = a\left[-\sin t + \frac{d}{dt}(t)\cdot\sin t + t\frac{d}{dt}(\sin t)\right]$$

[by using product rule of derivative]

$$\Rightarrow \frac{dx}{dt} = a(-\sin t + 1\cdot\sin t + t\cos t) = at\cos t \quad \text{...(i)} \quad \textbf{(1)}$$

Also given, $y = a(\sin t - t\cos t)$

On differentiating both sides w.r.t. t, we get

$$\frac{dy}{dt}=a[\cos t-\frac{d}{dt}(t)\cos t-t\frac{d}{dt}(\cos t)]$$

[by using product rule of derivative]

$$\frac{dy}{dt}=a(\cos t-\cos t\cdot 1+t\sin t)=at\sin t \quad \text{...(ii) (1)}$$

Now, $$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{at\sin t}{at\cos t}=\tan t$$

[from Eqs. (i) and (ii)]

Again, differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dt}(\tan t)\frac{dt}{dx}=\sec^2 t\frac{1}{dx/dt}$$

$$=\frac{\sec^2 t}{at\cos t}=\frac{\sec^3 t}{at} \quad \text{[from Eq. (i)] (1)}$$

Also, $$\frac{d^2x}{dt^2}=\frac{d}{dt}(at\cos t)$$

$$=a\frac{d}{dt}(t\cos t)$$

$$=a\left[\frac{d}{dt}(t)\cdot\cos t+t\frac{d}{dt}(\cos t)\right]$$

[by using product rule of derivative]

$$=a[\cos t-t\sin t]$$

and $$\frac{d^2y}{dt^2}=\frac{d}{dt}\left(\frac{dy}{dt}\right)=\frac{d}{dt}(at\sin t)$$

$$=a(\sin t+t\cos t) \quad \text{(1)}$$

78. Do same as Q. No. 19.

$$\left[\begin{array}{l}\textbf{Ans. } \frac{d^2y}{dx^2}=\frac{\sin t\sec^4 t}{a}\\ \text{Also, } \frac{d^2y}{dt^2}=\frac{d}{dt}\left(\frac{dy}{dt}\right)=\frac{d}{dt}(a\cos t)=-a\sin t\end{array}\right]$$

79. Given, $$y=x^{\cot x}+\frac{2x^2-3}{x^2+x+2}$$

Let $u=x^{\cot x}$ and $v=\dfrac{2x^2-3}{x^2+x+2}$

Then, given equation becomes

$$y=u+v$$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \text{ [differentiate w.r.t. } x]\text{...(i) (1/2)}$$

Consider $u=x^{\cot x}$

On taking log both sides, we get

$$\log u=\cot x\log x$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{u}\frac{du}{dx}=\cot x\cdot\frac{1}{x}-\operatorname{cosec}^2 x\cdot\log x$$

$$\Rightarrow \frac{du}{dx}=u\left(\frac{\cot x}{x}-\operatorname{cosec}^2 x\cdot\log x\right)$$

$$=x^{\cot x}\left(\frac{\cot x}{x}-\operatorname{cosec}^2 x\cdot\log x\right)\text{...(ii) (1)}$$

Now, consider $v=\dfrac{2x^2-3}{x^2+x+2}$

On differentiating both sides w.r.t. x, we get

$$\frac{dv}{dx}=\frac{(x^2+x+2)(4x)-(2x^2-3)(2x+1)}{(x^2+x+2)^2}$$

[by using quotient rule of derivative]

$$=\frac{4x^3+4x^2+8x-4x^3-2x^2+6x+3}{(x^2+x+2)^2}$$

$$=\frac{2x^2+14x+3}{(x^2+x+2)^2} \quad \text{...(iii) (1½)}$$

On substituting the values from Eqs. (ii) and (iii) to Eq. (i), we get

$$\frac{dy}{dx}=x^{\cot x}\left(\frac{\cot x}{x}-\operatorname{cosec}^2 x\cdot\log x\right)+\frac{2x^2+14x+3}{(x^2+x+2)^2} \quad \text{(1)}$$

80. Given, $x=\tan\left(\frac{1}{a}\log y\right)$

$$\Rightarrow \tan^{-1}x=\frac{1}{a}\log y \quad [\because \tan\theta=a\Rightarrow\theta=\tan^{-1}a]$$

$$\Rightarrow a\tan^{-1}x=\log y$$

On differentiating both sides w.r.t. x, we get

$$a\times\frac{1}{1+x^2}=\frac{1}{y}\cdot\frac{dy}{dx}$$

$$\left[\because \frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}\right] \quad \text{(1)}$$

$$\Rightarrow (1+x^2)\frac{dy}{dx}=ay \quad \text{(1)}$$

Again, differentiating both sides w.r.t. x, we get

$$(1+x^2)\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right)+\frac{dy}{dx}\cdot\frac{d}{dx}(1+x^2)=\frac{d}{dx}(ay)$$

[by using product rule of derivative] (1)

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2}+\frac{dy}{dx}\cdot(2x)=a\cdot\frac{dy}{dx}$$

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-a\frac{dy}{dx}=0$$

$$\therefore (1+x^2)\frac{d^2y}{dx^2}+(2x-a)\frac{dy}{dx}=0 \quad \text{(1)}$$

Hence proved.

81. Do same as Q. No. 76.

$$\left[\text{Ans. } x^{x\cos x}\,[\cos x - x\log x\sin x + \log x\cos x] - \frac{4x}{(x^2-1)^2}\right]$$

82. Given, $x = a(\theta - \sin\theta)$ and $y = a(1+\cos\theta)$

On differentiating both sides of x and y w.r.t. θ, we get

$$\frac{dx}{d\theta} = a(1-\cos\theta) \text{ and } \frac{dy}{d\theta} = -a\sin\theta \quad (1)$$

$$\therefore \quad \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{-a\sin\theta}{a(1-\cos\theta)} = \frac{-\sin\theta}{1-\cos\theta}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = -\cot\frac{\theta}{2} \quad \ldots(\text{i})\ (1)$$

$$\left[\because \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \text{ and } 1-\cos\theta = 2\sin^2\frac{\theta}{2}\right]$$

Again, differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-\cot\frac{\theta}{2}\right)$$

$$= \frac{d}{d\theta}\left(-\cot\frac{\theta}{2}\right)\times\frac{d\theta}{dx} \qquad \left[\because \frac{d}{dx}[f(\theta)] = \frac{d}{d\theta}f(\theta)\times\frac{d\theta}{dx}\right]$$

$$= \frac{1}{2}\operatorname{cosec}^2\frac{\theta}{2}\times\frac{1}{a(1-\cos\theta)} \quad (1)$$

$$\left[\because \frac{d}{d\theta}(\cot\theta) = -\operatorname{cosec}^2\theta\right]$$

$$= \frac{1}{2a}\operatorname{cosec}^2\frac{\theta}{2}\times\frac{1}{2\sin^2\frac{\theta}{2}} = \frac{1}{4a}\operatorname{cosec}^4\frac{\theta}{2} \quad (1)$$

83. We have, LHS $= \dfrac{d}{dx}\left[\dfrac{x}{2}\sqrt{a^2-x^2} + \dfrac{a^2}{2}\sin^{-1}\dfrac{x}{a}\right]$

$$= \left[\frac{x}{2}\times\frac{d}{dx}\sqrt{a^2-x^2} + \sqrt{a^2-x^2}\times\frac{d}{dx}\left(\frac{x}{2}\right) + \frac{a^2}{2}\times\frac{d}{dx}\sin^{-1}\frac{x}{a}\right] (1)$$

[by using product rule of derivative]

$$= \frac{x}{2}\cdot\frac{1}{2\sqrt{a^2-x^2}}\frac{d}{dx}(a^2-x^2) + \sqrt{a^2-x^2}\cdot\frac{1}{2} + \frac{a^2}{2}\cdot\frac{1}{\sqrt{1-\frac{x^2}{a^2}}}\cdot\frac{d}{dx}\left(\frac{x}{a}\right)$$

[by using chain rule of derivative]

$$= \left[\frac{x}{2}\cdot\frac{-2x}{2\sqrt{a^2-x^2}} + \sqrt{a^2-x^2}\cdot\frac{1}{2} + \frac{a^2}{2}\times\frac{1}{\sqrt{1-\frac{x^2}{a^2}}}\cdot\frac{1}{a}\right] (1)$$

$$= \frac{-x^2}{2\sqrt{a^2-x^2}} + \frac{\sqrt{a^2-x^2}}{2} + \frac{a^2}{2a}\cdot\frac{1}{\sqrt{\frac{a^2-x^2}{a^2}}}$$

$$= \frac{-x^2}{2\sqrt{a^2-x^2}} + \frac{\sqrt{a^2-x^2}}{2} + \frac{a^2}{2a}\times\frac{a}{\sqrt{a^2-x^2}}$$

$$= \frac{-x^2}{2\sqrt{a^2-x^2}} + \frac{\sqrt{a^2-x^2}}{2} + \frac{a^2}{2\sqrt{a^2-x^2}} \quad (1)$$

$$= \frac{-x^2 + (a^2-x^2) + a^2}{2\sqrt{a^2-x^2}}$$

$$= \frac{2a^2-2x^2}{2\sqrt{a^2-x^2}} = \frac{2(a^2-x^2)}{2\sqrt{a^2-x^2}} = \sqrt{a^2-x^2}$$

= RHS **Hence proved.** (1)

84. Do same as Q. No. 63.

85. Given, $\log(\sqrt{1+x^2} - x) = y\sqrt{1+x^2}$ ...(i)

On differentiating both sides w.r.t. x, we get

$$\frac{1}{\sqrt{1+x^2}-x}\frac{d}{dx}[\sqrt{1+x^2}-x] = y\frac{d}{dx}\sqrt{1+x^2} + \sqrt{1+x^2}\frac{dy}{dx} \quad (1)$$

[by using chain rule and product rule of derivative]

$$\Rightarrow \frac{1}{\sqrt{1+x^2}-x}\left[\frac{1}{2\sqrt{1+x^2}}\frac{d}{dx}(1+x^2) - 1\right] = \frac{y}{2\sqrt{1+x^2}}\frac{d}{dx}(1+x^2) + \sqrt{1+x^2}\frac{dy}{dx}$$

$$\Rightarrow \frac{1}{\sqrt{1+x^2}-x}\left[\frac{2x}{2\sqrt{1+x^2}} - 1\right] = y\times\frac{2x}{2\sqrt{1+x^2}} + \sqrt{1+x^2}\cdot\frac{dy}{dx} \quad (1½)$$

$$\Rightarrow \frac{1}{\sqrt{1+x^2}-x}\left[\frac{x-\sqrt{1+x^2}}{\sqrt{1+x^2}}\right] = \frac{xy}{\sqrt{1+x^2}} + \sqrt{1+x^2}\cdot\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{-1}{\sqrt{1+x^2}} = \frac{xy + (1+x^2)\frac{dy}{dx}}{\sqrt{1+x^2}}$$

$$\Rightarrow \quad -1 = xy + (1+x^2)\frac{dy}{dx}$$

$$\therefore \quad (1+x^2)\frac{dy}{dx} + xy + 1 = 0 \quad \textbf{(1½)}$$

Hence proved.

86. Do same as Q. No. 82. $\left[\text{Ans. } \frac{1}{4a}\sec^4\frac{\theta}{2}\right]$

87. First, we differentiate the given expression with respect to x and get first derivative of y. Then, put the value of y and first derivative of y in LHS of given expression and then solve it to get the required RHS.

To prove $\quad y^2 + \left(\frac{dy}{dx}\right)^2 = a^2 + b^2 \quad$ ...(i)

Given, $y = a \sin x + b \cos x \quad$ ...(ii)

On differentiating both sides of Eq. (ii) w.r.t. x, we get

$$\frac{dy}{dx} = a \cos x - b \sin x \quad \textbf{(1)}$$

Now, let us take LHS of Eq. (i).

Here, $\quad \text{LHS} = y^2 + \left(\frac{dy}{dx}\right)^2$

On putting the value of y and dy/dx, we get

$\text{LHS} = (a \sin x + b \cos x)^2 + (a \cos x - b \sin x)^2$

$= a^2 \sin^2 x + b^2 \cos^2 x + 2ab \sin x \cos x$

$\quad + a^2 \cos^2 x + b^2 \sin^2 x - 2ab \sin x \cos x$ **(1½)**

$= a^2 \sin^2 x + b^2 \cos^2 x + a^2 \cos^2 x + b^2 \sin^2 x$

$= a^2 (\sin^2 x + \cos^2 x) + b^2 (\sin^2 x + \cos^2 x)$

$= a^2 + b^2 \quad [\because \sin^2 x + \cos^2 x = 1]$

$= \text{RHS}$ **(1½)**

Hence proved.

88. Do same as Q. No. 56. $\left[\text{Ans. } \frac{\sec^3\theta}{a\theta}\right]$

89. Given, $\quad x = a(\theta - \sin\theta)$

and $\quad y = a(1 + \cos\theta)$

On differentiating both sides w.r.t. θ, we get

$$\frac{dx}{d\theta} = a(1 - \cos\theta) \quad \text{and} \quad \frac{dy}{d\theta} = -a \sin\theta \quad \textbf{(1)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-a \sin\theta}{a(1-\cos\theta)} \quad \textbf{(1)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2a \sin\frac{\theta}{2}\cos\frac{\theta}{2}}{a \times 2\sin^2\frac{\theta}{2}}$$

$$\left[\because \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} \text{ and } 1 - \cos x = 2\sin^2\frac{x}{2}\right] \quad \textbf{(1)}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\cot\frac{\theta}{2}$$

On putting $\theta = \frac{\pi}{3}$, we get

$$\left[\frac{dy}{dx}\right]_{\theta = \frac{\pi}{3}} = -\cot\frac{\pi}{6} = -\sqrt{3} \quad \left[\because \cot\frac{\pi}{6} = \sqrt{3}\right]$$

Hence, $\frac{dy}{dx}$ at $\theta = \frac{\pi}{3}$ is $-\sqrt{3}$. **(1)**

90. First, take log on both sides and then differentiate to get the required value of $\frac{dy}{dx}$.

Given, $y = (\sin x - \cos x)^{(\sin x - \cos x)}, \frac{\pi}{4} < x < \frac{3\pi}{4}$

On taking log both sides, we get

$$\log y = \log (\sin x - \cos x)^{(\sin x - \cos x)}$$

$$\Rightarrow \quad \log y = (\sin x - \cos x) \cdot \log (\sin x - \cos x)$$

$$[\because \log m^n = n \log m] \quad \textbf{(1)}$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = (\sin x - \cos x) \times \frac{d}{dx}\log(\sin x - \cos x) + \log(\sin x - \cos x) \times \frac{d}{dx}(\sin x - \cos x)$$

[by using product rule of derivative]

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = (\sin x - \cos x)\cdot\frac{1}{(\sin x - \cos x)}\cdot\frac{d}{dx}(\sin x - \cos x) + \log(\sin x - \cos x)\cdot(\cos x + \sin x) \quad \textbf{(1)}$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = (\sin x - \cos x)\frac{1}{(\sin x - \cos x)}(\cos x + \sin x) + \log(\sin x - \cos x)\cdot(\cos x + \sin x)$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = (\cos x + \sin x) + (\cos x + \sin x)\cdot\log(\sin x - \cos x) \quad \textbf{(1)}$$

$$\Rightarrow \frac{dy}{dx} = y(\cos x + \sin x)[1 + \log(\sin x - \cos x)]$$

$$\therefore \frac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)} \cdot (\cos x + \sin x)[1 + \log(\sin x - \cos x)]$$

(1)

91. In the given expression, put $x = \sin\theta$ and simplify the resulting expression, then differentiate it.

Given, $y = \cos^{-1}\left[\frac{2x - 3\sqrt{1-x^2}}{\sqrt{13}}\right]$

Put $x = \sin\theta$, then $\theta = \sin^{-1} x$ **(1/2)**

$$\therefore y = \cos^{-1}\left[\frac{2\sin\theta - 3\sqrt{1-\sin^2\theta}}{\sqrt{13}}\right]$$

$$\Rightarrow y = \cos^{-1}\left[\frac{2\sin\theta - 3\cos\theta}{\sqrt{13}}\right]$$

$[\because \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta]$

$$\Rightarrow y = \cos^{-1}\left[\frac{2}{\sqrt{13}}\sin\theta - \frac{3}{\sqrt{13}}\cos\theta\right]$$ **(1/2)**

Now, let $\frac{2}{\sqrt{13}} = \cos\alpha$, then $\frac{3}{\sqrt{13}} = \sin\alpha$

$$\left[\because \sin^2\alpha + \cos^2\alpha = \left(\frac{3}{\sqrt{13}}\right)^2 + \left(\frac{2}{\sqrt{13}}\right)^2 = \frac{9}{13} + \frac{4}{13} = \frac{13}{13} = 1\right]$$

$$\therefore y = \cos^{-1}[\sin\theta\cos\alpha - \cos\theta\sin\alpha]$$

$$\Rightarrow y = \cos^{-1}[\sin(\theta - \alpha)]$$ **(1)**

$[\because \sin\theta\cos\alpha - \cos\theta\sin\alpha = \sin(\theta-\alpha)]$

$$\Rightarrow y = \cos^{-1}\cos\left[\frac{\pi}{2} - (\theta - \alpha)\right]$$

$$\Rightarrow y = \frac{\pi}{2} - \theta + \alpha \quad [\because \cos^{-1}(\cos\theta) = \theta]$$

$$\Rightarrow y = \frac{\pi}{2} - \sin^{-1} x + \alpha \quad [\because \theta = \sin^{-1} x]$$ **(1)**

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 0 - \frac{1}{\sqrt{1-x^2}} + 0\left[\because \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}\right]$$

$$\therefore \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$$ **(1)**

92. Do same as Q. No. 75.

93. Given, $y = \text{cosec}^{-1} x$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{-1}{x\sqrt{x^2-1}} \quad [\because x > 1, \text{ given}]$$

$$\Rightarrow x\sqrt{x^2-1} \cdot \frac{dy}{dx} = -1$$ **(1)**

Again, differentiating both sides w.r.t. x, we get

$$(x\sqrt{x^2-1}) \cdot \frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx} \cdot \frac{d}{dx}(x\sqrt{x^2-1}) = \frac{d}{dx}(-1)$$

[by using product rule of derivative] **(1)**

$$\Rightarrow x\sqrt{x^2-1} \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx}\left\{x \times \frac{d}{dx}\sqrt{x^2-1} + \sqrt{x^2-1} \times \frac{d}{dx}(x)\right\} = 0$$

$$\Rightarrow x\sqrt{x^2-1}\frac{d^2y}{dx^2} + \frac{dy}{dx}\left\{\frac{x}{2\sqrt{x^2-1}}\frac{d}{dx}(x^2-1) + \sqrt{x^2-1} \times 1\right\} = 0$$

[by using chain rule of derivative]

$$\Rightarrow x\sqrt{x^2-1}\frac{d^2y}{dx^2} + \frac{dy}{dx}\left\{\frac{x \cdot 2x}{2\sqrt{x^2-1}} + \sqrt{x^2-1}\right\} = 0$$

$$\Rightarrow x\sqrt{x^2-1}\frac{d^2y}{dx^2} + \frac{dy}{dx}\left\{\frac{x^2}{\sqrt{x^2-1}} + \sqrt{x^2-1}\right\} = 0$$

$$\Rightarrow x\sqrt{x^2-1}\frac{d^2y}{dx^2} + \frac{dy}{dx}\left\{\frac{x^2 + x^2 - 1}{\sqrt{x^2-1}}\right\} = 0$$ **(1)**

$$\Rightarrow x(x^2-1)\frac{d^2y}{dx^2} + (x^2 + x^2 - 1)\frac{dy}{dx} = 0$$

$$\therefore x(x^2-1)\frac{d^2y}{dx^2} + (2x^2 - 1)\frac{dy}{dx} = 0$$ **(1)**

Hence proved.

94. Do same as Q. No. 91. $\left[\text{Ans. } \frac{1}{\sqrt{1-x^2}}\right]$

95. Given function is $f(x) = \begin{cases} 3x - 2, & 0 < x \le 1 \\ 2x^2 - x, & 1 < x \le 2 \\ 5x - 4, & x > 2 \end{cases}$

First, we show the continuity of above function at $x = 1$ and at $x = 2$

Continuity at $x = 1$

$$\text{LHL} = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (3x - 2)$$

$\Rightarrow$ LHL $= \lim_{h \to 0} [3(1-h) - 2]$

[put $x = 1 - h$; when $x \to 1^-$, then $h \to 0$]

$= \lim_{h \to 0} (3 - 3h - 2) = 1$ **(1/2)**

RHL $= \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2x^2 - x)$

[put $x = 1 + h$; when $x \to 1^+$, then $h \to 0$]

$\Rightarrow$ RHL $= \lim_{h \to 0} [2(1+h)^2 - (1+h)]$

$= \lim_{h \to 0} [2(1 + h^2 + 2h) - (1+h)]$

$= \lim_{h \to 0} [2 + 2h^2 + 4h - 1 - h]$

$= \lim_{h \to 0} (2h^2 + 3h + 1)$

$\Rightarrow$ RHL $= 1$

Also, $f(1) = 3(1) - 2 = 3 - 2 = 1$

Since, LHL = RHL = $f(1)$

$\therefore$ $f(x)$ is continuous at $x = 1$. **(1/2)**

Continuity at $x = 2$

LHL $= \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2x^2 - x)$

$\Rightarrow$ LHL $= \lim_{h \to 0} [2(2-h)^2 - (2-h)]$

[put $x = 2 - h$; when $x \to 2^-$, then $h \to 0$]

$= \lim_{h \to 0} [2(4 + h^2 - 4h) - (2-h)]$

$= \lim_{h \to 0} (8 + 2h^2 - 8h - 2 + h)$

$\Rightarrow$ LHL $= 8 - 2 = 6$ [put $h = 0$] **(1/2)**

and RHL $= \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (5x - 4)$

$\Rightarrow$ RHL $= \lim_{h \to 0} [5(2+h) - 4]$

[put $x = 2 + h$; when $x \to 2^+$, then $h \to 0$]

$= \lim_{h \to 0} (10 + 5h - 4) = \lim_{h \to 0} (5h + 6)$

$\Rightarrow$ RHL $= 6$

Also, $f(2) = 2(2)^2 - 2 = 8 - 2 = 6$

Since, LHL = RHL = $f(2)$

$\therefore$ $f(x)$ is continuous at $x = 2$ **(1/2)**

Hence, $f(x)$ is continuous at all indicated points. **(1/2)**

Now, let us verify differentiability of the given function at $x = 2$

Differentiability at $x = 2$

LHD $= \lim_{h \to 0} \dfrac{f(2-h) - f(2)}{-h}$

$\Rightarrow$ LHD $= \lim_{h \to 0} \dfrac{[2(2-h)^2 - (2-h)] - [8-2]}{-h}$

$= \lim_{h \to 0} \dfrac{2(4 + h^2 - 4h) - (2-h) - 6}{-h}$

$= \lim_{h \to 0} \dfrac{8 + 2h^2 - 8h - 2 + h - 6}{h}$

$= \lim_{h \to 0} \dfrac{h(2h-7)}{-h} = \lim_{h \to 0} -(2h-7)$ **(1/2)**

$\Rightarrow$ LHD $= 7$

and RHD $= \lim_{h \to 0} \dfrac{f(2+h) - f(2)}{h}$

$= \lim_{h \to 0} \dfrac{[5(2+h) - 4] - [8-2]}{h}$

$= \lim_{h \to 0} \dfrac{(6 + 5h) - (6)}{h} = \lim_{h \to 0} \dfrac{5h}{h}$

$\Rightarrow$ RHD $= 5$

Since, LHD $\neq$ RHD **(1/2)**

So, $f(x)$ is not differentiable at $x = 2$

Hence, $f(x)$ is continuous at $x = 1$ and $x = 2$ but not differentiable at $x = 2$ **(1/2)**

Hence proved.

96. Do same as Q. No. 38.

97. Given, $y = (\cos x)^x + (\sin x)^{1/x}$

Let $u = (\cos x)^x$ and $v = (\sin x)^{1/x}$

Then, given equation becomes

$$y = u + v$$

On differentiating both sides w.r.t. x, we get

$\Rightarrow$ $\dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$ ...(i) **(1/2)**

Consider, $u = (\cos x)^x$

On taking log both sides, we get

$\log u = \log (\cos x)^x$

$\Rightarrow$ $\log u = x \log (\cos x)$

[$\because \log m^n = n \log m$] **(1/2)**

On differentiating both sides w.r.t. x, we get

$\dfrac{1}{u}\dfrac{du}{dx} = x \cdot \dfrac{d}{dx} \log (\cos x) + \log (\cos x) \cdot \dfrac{d}{dx}(x)$

[by using product rule of derivative]

$\Rightarrow$ $\dfrac{1}{u}\dfrac{du}{dx} = x \cdot \dfrac{1}{\cos x}(-\sin x) + \log \cos x \cdot 1$

$\Rightarrow$ $\dfrac{1}{u}\dfrac{du}{dx} = -x \tan x + \log (\cos x)$

$\Rightarrow \quad \frac{du}{dx} = u[-x \tan x + \log \cos x]$

$\Rightarrow \quad \frac{du}{dx} = (\cos x)^x [-x \tan x + \log \cos x]$ ...(ii) (1)

Now, consider $v = (\sin x)^{1/x}$

On taking log both sides, we get

$$\log v = \log (\sin x)^{1/x}$$

$$\Rightarrow \quad \log v = \frac{1}{x} \log \sin x$$

[$\because \log m^n = n \log m$] (1/2)

On differentiating both sides w.r.t. x, we get

$$\frac{1}{v} \cdot \frac{dv}{dx} = \frac{1}{x} \cdot \frac{d}{dx} (\log \sin x) + \log \sin x \cdot \frac{d}{dx}\left(\frac{1}{x}\right)$$

$$\Rightarrow \frac{1}{v} \cdot \frac{dv}{dx} = \frac{1}{x} \cdot \frac{1}{\sin x} \cdot \cos x + \log \sin x \left(-\frac{1}{x^2}\right)$$

$$\Rightarrow \frac{1}{v} \cdot \frac{dv}{dx} = \frac{\cot x}{x} - \frac{\log (\sin x)}{x^2}$$

$$\Rightarrow \frac{dv}{dx} = v\left(\frac{\cot x}{x} - \frac{\log (\sin x)}{x^2}\right)$$

$$\Rightarrow \frac{dv}{dx} = (\sin x)^{1/x}\left[\frac{\cot x}{x} - \frac{\log (\sin x)}{x^2}\right] \quad \text{...(iii)}$$

(1/2)

Now, from Eqs. (i), (ii) and (iii), we get

$$\frac{dy}{dx} = (\cos x)^x [-x \tan x + \log \cos x]$$

$$+ (\sin x)^{1/x}\left[\frac{\cot x}{x} - \frac{\log (\sin x)}{x^2}\right] \text{ (1)}$$

98. First, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, and then put their values along with value of y in LHS of proven expression.

To prove $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$

Given, $y = e^x \sin x$ ...(i)

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = e^x \cdot \frac{d}{dx} (\sin x) + \sin x \cdot \frac{d}{dx} (e^x)$$

[by using product rule of derivative]

$\Rightarrow \quad \frac{dy}{dx} = e^x \cdot \cos x + \sin x \cdot e^x$ (1)

$\Rightarrow \quad \frac{dy}{dx} = e^x (\cos x + \sin x)$ ...(ii)

Again, differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = e^x \cdot \frac{d}{dx} (\cos x + \sin x)$$

$$+ (\cos x + \sin x) \cdot \frac{d}{dx} (e^x)$$

[by using product rule of derivative]

$$\Rightarrow \quad \frac{d^2y}{dx^2} = e^x (-\sin x + \cos x)$$

$$+ (\cos x + \sin x) \cdot e^x$$

$$= e^x [-\sin x + \cos x + \cos x + \sin x]$$

$\Rightarrow \quad \frac{d^2y}{dx^2} = 2\cos x \, e^x$ ...(iii) (1)

Now, consider

$\text{LHS} = \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y$ (1)

$$= 2e^x \cos x - 2e^x (\cos x + \sin x) + 2e^x \sin x$$

[from Eqs. (i), (ii) and (iii)]

$$= 2e^x \cos x - 2e^x \cos x - 2e^x \sin x + 2e^x \sin x$$

$= 0 = \text{RHS}$ (1) **Hence proved.**

99. Given, $y = (x)^x + (\sin x)^x$

Let $u = (x)^x$

and $v = (\sin x)^x$

Then, given equation becomes, $y = u + v$

On differentiating both sides w.r.t. x, we get

$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ...(i) (1/2)

Consider, $u = x^x$

On taking log both sides, we get

$$\log u = \log x^x$$

$\Rightarrow \quad \log u = x \log x$ [$\because \log m^n = n \log m$] (1/2)

On differentiating both sides w.r.t. x, we get

$$\frac{1}{u} \frac{du}{dx} = x \cdot \frac{d}{dx} (\log x) + \log x \cdot \frac{d}{dx} (x)$$

[by using product rule of derivative]

$$\Rightarrow \quad \frac{1}{u} \frac{du}{dx} = x \cdot \frac{1}{x} + \log x \cdot 1$$

$$\Rightarrow \quad \frac{1}{u} \frac{du}{dx} = 1 + \log x$$

$$\Rightarrow \quad \frac{du}{dx} = u (1 + \log x)$$

$\Rightarrow \quad \frac{du}{dx} = x^x (1 + \log x)$ [put $u = x^x$] ...(ii) (1)

Now, consider, $v = (\sin x)^x$

On taking log both sides, we get

$$\log v = \log (\sin x)^x$$

$\Rightarrow \quad \log v = x \log (\sin x)$

[$\because \log m^n = n \log m$] (1/2)

On differentiating both sides w.r.t. x, we get

$$\frac{1}{v}\frac{dv}{dx} = x \cdot \frac{d}{dx} \log (\sin x) + \log (\sin x) \cdot \frac{d}{dx}(x)$$

[by using product rule of derivative]

$\Rightarrow \quad \frac{1}{v}\frac{dv}{dx} = x \cdot \frac{1}{\sin x} \cdot \frac{d}{dx}(\sin x) + \log \sin x$

$\Rightarrow \quad \frac{1}{v}\frac{dv}{dx} = x \cdot \frac{1}{\sin x} \cdot \cos x + \log \sin x$

$\left[\because \frac{d}{dx}(\log v) = \frac{1}{v}\frac{dv}{dx}\right]$

$\Rightarrow \quad \frac{1}{v}\frac{dv}{dx} = x \cot x + \log \sin x$

$\Rightarrow \quad \frac{dv}{dx} = v\,(x \cot x + \log \sin x)$

$\Rightarrow \quad \frac{dv}{dx} = (\sin x)^x\,(x \cot x + \log \sin x)$ ...(iii) (1)

[put $v = (\sin x)^x$]

Now, from Eqs. (i), (ii) and (iii), we get

$$\frac{dy}{dx} = x^x\,(1 + \log x)$$
$$+ (\sin x)^x\,(x \cot x + \log \sin x) \text{ (1/2)}$$

Objective Questions

(For Complete Chapter)

1 Mark Questions

1. If $f(x) = \begin{cases} \frac{3 \sin \pi x}{5x}, & x \neq 0 \\ 2k, & x = 0 \end{cases}$ is continuous at $x = 0$, then the value of k is

(a) $\frac{\pi}{10}$ (b) $\frac{3\pi}{10}$
(c) $\frac{3\pi}{2}$ (d) $\frac{3\pi}{5}$

2. If $f(x) = \begin{cases} ax + 3, & x \leq 2 \\ a^2x - 1, & x > 2 \end{cases}$, then the values of a for which f is continuous for all x are

(a) 1 and -2 (b) 1 and 2
(c) -1 and 2 (d) -1 and -2

3. Discuss the continuity of the function $f(x) = \sin 2x - 1$ at the point $x = 0$ and $x = \pi$

(a) Continuous at $x = 0, \pi$
(b) Discontinuous at $x = 0$ but continuous at $x = \pi$
(c) Continuous at $x = 0$ but discontinuous at $x = \pi$
(d) Discontinuous at $x = 0, \pi$

4. If $f(x) = \begin{cases} \frac{\log x}{x - 1}, & \text{if } x \neq 1 \\ k, & \text{if } x = 1 \end{cases}$ is continuous at $x = 1$, then the value of k is

(a) 0 (b) -1 (c) 1 (d) e

5. The points of discontinuity of $\tan x$ are

(a) $n\pi, n \in I$ (b) $2n\pi, n \in I$
(c) $(2n + 1)\frac{\pi}{2}, n \in I$ (d) None of these

6. At $x = \frac{3}{2}$, the function $f(x) = \frac{|2x - 3|}{2x - 3}$ is

(a) continuous (b) discontinuous
(c) differentiable (d) non-zero

7. If $f(x) = \frac{x}{1 + x}$ and $g(x) = f\,[f(x)]$, then $g'(x)$ is equal to

(a) $\frac{1}{(2x + 3)^2}$ (b) $\frac{1}{(x + 1)^2}$
(c) $\frac{1}{x^2}$ (d) $\frac{1}{(2x + 1)^2}$

8. If $\sin (x + y) = \log (x + y)$, then dy/dx is equal to

(a) 2 (b) -2
(c) 1 (d) -1

9. If $x = e^{y + e^{y + e^{y + \cdots}}}$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{1}{x}$ (b) $\frac{1 - x}{x}$
(c) $\frac{x}{1 + x}$ (d) None of these

10. If x is measured in degrees, then $\frac{d}{dx}(\cos x)$ is equal to

(a) $-\sin x$ (b) $-\frac{180}{\pi}\sin x$

(c) $-\frac{\pi}{180}\sin x$ (d) $\sin x$

11. If $f(x) = \log_e(\log_e x)$, then $f'(e)$ is equal to

(a) e^{-1} (b) e (c) 1 (d) 0

12. The derivative of $\cos^3 x$ with respect to $\sin^3 x$ is

(a) $-\cot x$ (b) $\cot x$ (c) $\tan x$ (d) $-\tan x$

13. If $y = \tan^{-1}\sqrt{\frac{1-\sin x}{1+\sin x}}$, then the value of $\frac{dy}{dx}$ at $x = \frac{\pi}{6}$ is

(a) $-\frac{1}{2}$ (b) $\frac{1}{2}$ (c) 1 (d) -1

14. If $y = \log[\sin(x^2)], 0 < x < \frac{\pi}{2}$, then $\frac{dy}{dx}$ at $x = \frac{\sqrt{\pi}}{2}$ is

(a) 0 (b) 1 (c) $\pi/4$ (d) $\sqrt{\pi}$

15. The derivative of $\log|x|$ is

(a) $\frac{1}{x}, x > 0$ (b) $\frac{1}{|x|}, x \neq 0$

(c) $\frac{1}{x}, x \neq 0$ (d) None of these

16. If $x = \frac{2at}{1+t^3}$ and $y = \frac{2at^2}{(1+t^3)^2}$, then $\frac{dy}{dx}$ is equal to

(a) ax (b) a^2x^2 (c) $\frac{x}{a}$ (d) $\frac{x}{2a}$

Solutions

1. (*b*) Given, $f(x) = \begin{cases} \frac{3\sin \pi x}{5x}, & x \neq 0 \\ 2k, & x = 0 \end{cases}$

Now, $\lim_{x \to 0} f(x) = \lim_{x \to 0}\left(\frac{3\sin \pi x}{5x}\right)$

$= \frac{3}{5}\lim_{x \to 0}\left(\frac{\sin \pi x}{\pi x}\right) \times \pi = \frac{3}{5} \times 1 \times \pi = \frac{3}{5}\pi$

Also, $f(0) = 2k$

Since, $f(x)$ is continuous at $x = 0$,

$\therefore\ f(0) = \lim_{x \to 0} f(x) \Rightarrow 2k = \frac{3}{5}\pi \Rightarrow k = \frac{3\pi}{10}$

2. (*c*) Given, $f(x) = \begin{cases} ax + 3, & x \leq 2 \\ a^2x - 1, & x > 2 \end{cases}$

Continuity at $x = 2$,

LHL $= \lim_{x \to 2^-} f(x) = \lim_{x \to 2}(ax + 3) = 2a + 3$

RHL $= \lim_{x \to 2^+} f(x) = \lim_{x \to 2}(a^2x - 1) = 2a^2 - 1$

Since, $f(x)$ is continuous for all values of x.

$\therefore$ LHL = RHL

$\Rightarrow 2a + 3 = 2a^2 - 1$

$\Rightarrow 2a^2 - 2a - 4 = 0$

$\Rightarrow a^2 - a - 2 = 0$ [dividing each term by 2]

$\Rightarrow a^2 - 2a + a - 2 = 0$

$\Rightarrow a(a-2) + 1(a-2) = 0 \Rightarrow (a+1)(a-2) = 0$

$\therefore a = -1, 2$

3. (*a*) $f(x) = \sin 2x - 1$

Being sine function, $f(x)$ is defined and continuous at $x = 0, \pi$.

4. (*c*) At $x = 1$,

LHL $= \lim_{x \to 1^+} f(x) = \lim_{h \to 0}\frac{\log(1+h)}{1+h-1}$

$= \lim_{h \to 0}\frac{\log(1+h)}{h} = 1$

As $f(x)$ is continuous at $x = 1$.

$\therefore \lim_{x \to 1^+} f(x) = f(1) \Rightarrow 1 = k$

5. (*c*) The points of discontinuity of $f(x)$ are those points, where $\tan x$ is infinite.

i.e. $\tan x = \tan\infty \Rightarrow x = (2n+1)\frac{\pi}{2}, n \in I$

6. (*b*) The given function $f(x)$ can be written as

$$f(x) = \begin{cases} \frac{2x-3}{2x-3}, & \text{if } x > \frac{3}{2} \\ \frac{-(2x-3)}{2x-3}, & \text{if } x < \frac{3}{2} \end{cases}$$

$$= \begin{cases} 1, & \text{if } x > \frac{3}{2} \\ -1, & \text{if } x < \frac{3}{2} \end{cases}$$

Now, RHL $= \lim_{x \to 3/2^+} f(x) = \lim_{x \to 3/2^+} 1 = 1$

and LHL $= \lim_{x \to 3/2^-} f(x) = \lim_{x \to 3/2^-} (-1) = -1$

$\because$ RHL $\neq$ LHL

So, $f(x)$ is discontinuous at $x = \frac{3}{2}$.

7. (*d*) Given, $f(x) = \frac{x}{1+x}$ and $g(x) = f[f(x)]$

$$\therefore g(x) = f\left(\frac{x}{x+1}\right) = \frac{\frac{x}{1+x}}{1 + \frac{x}{x+1}} \Rightarrow g(x) = \frac{x}{2x+1}$$

On differentiating both sides w.r.t. x, we get

$$g'(x) = \frac{(2x+1)1 - x(2)}{(2x+1)^2} = \frac{1}{(2x+1)^2}$$

8. (*d*) We have, $\sin(x+y) = \log(x+y)$

On differentiating w.r.t. x, we get

$$\cos(x+y)\left(1 + \frac{dy}{dx}\right) = \frac{1}{x+y}\left(1 + \frac{dy}{dx}\right)$$

$$\Rightarrow \left[\cos(x+y) - \frac{1}{x+y}\right]\left(1 + \frac{dy}{dx}\right) = 0$$

$$\therefore \frac{dy}{dx} = -1 \qquad \left[\because \cos(x+y) - \frac{1}{x+y} \neq 0\right]$$

9. (*b*) We have, $x = e^{y + e^{y + e^{y + \dots}}}$

$$x = e^{y+x}$$

On taking log both sides, we get

$$\log x = (x+y) \log e$$

$$\Rightarrow \frac{1}{x} = \frac{dy}{dx} + 1 \Rightarrow \frac{dy}{dx} = \frac{1-x}{x}$$

10. (c) $\frac{d}{dx}(\cos x) = -\frac{\pi}{180} \sin x$

11. (*a*) Given, $f(x) = \log_e(\log_e x)$

$$f'(x) = \frac{1}{x \log_e(x)} \Rightarrow f'(e) = \frac{1}{e}$$

12. (*a*) Let $u = \cos^3 x$ and $v = \sin^3 x$

$$\frac{du}{dx} = -3\cos^2 x \sin x \text{ and } \frac{dv}{dx} = 3\sin^2 x \cos x$$

$$\text{Now, } \frac{du}{dv} = \frac{-3\cos^2 x \sin x}{3\sin^2 x \cos x} = -\cot x$$

13. (*a*) Given, $y = \tan^{-1}\sqrt{\frac{1 - \sin x}{1 + \sin x}}$

$$= \tan^{-1}\sqrt{\frac{1 - \cos\left(\frac{\pi}{2} - x\right)}{1 + \cos\left(\frac{\pi}{2} - x\right)}}$$

$$= \tan^{-1}\left|\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right| = \frac{\pi}{4} - \frac{x}{2}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{2}$$

14. (*d*) Given, $y = \log[\sin(x^2)]$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sin x^2} \cdot \cos x^2 \cdot 2x = 2x \cot x^2$$

$$\text{At } x = \frac{\sqrt{\pi}}{2}, \frac{dy}{dx} = \frac{2\sqrt{\pi}}{2} \cot\left(\frac{\sqrt{\pi}}{2}\right)^2$$

$$= \sqrt{\pi} \cot\left(\frac{\pi}{4}\right) = \sqrt{\pi}$$

15. (c) We have, $y = \log|x| = \begin{cases} \log x, & x > 0 \\ \log(-x), & x < 0 \end{cases}$

$$\therefore \frac{dy}{dx} = \begin{cases} \frac{1}{x}, & x > 0 \\ \frac{1}{-x}(-1) = \frac{1}{x}, & x < 0 \end{cases}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x}, x \neq 0$$

16. (c) $\because x = \frac{2at}{1+t^3}$ and $y = \frac{2at^2}{(1+t^3)^2}$

$$\therefore 2ay = x^2 \Rightarrow \frac{dy}{dx} = \frac{x}{a}$$