DCAM classes

Dynamic Classes for Academic Mastery

Solutions

1. We have, $C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$

Clearly, the marginal cost, $MC(x) = \frac{d}{dx}C(x)$

$$= \frac{d}{dx}(0.005x^3 - 0.02x^2 + 30x + 5000)$$

$$= 0.005 \times 3x^2 - 0.02 \times 2x + 30 + 0$$

$$= 0.015x^2 - 0.04x + 30$$ **(1)**

Now, marginal cost when 3 units are produced

$$= MC(3) = 0.015(9) - 0.04(3) + 30$$

$$= 0.135 - 0.12 + 30 = 30.015$$ **(1)**

2. Marginal Revenue (MR) $= \frac{dR}{dx} = \frac{d}{dx}(3x^2 + 36x + 5)$

$$= 6x + 36$$ **(1)**

$\therefore$ When $x = 5$

Marginal Revenue (MR) $= 6 \times 5 + 36 = 66$ **(1)**

3. Let r be the radius, V be the volume and S be the surface area of sphere.

Then, we have $\frac{dV}{dt} = 8 \text{ cm}^3/\text{s}$

To find $\frac{dS}{dt}$, when $r = 12$ cm

Since, $V = \frac{4}{3}\pi r^3$

$\therefore \quad \frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt}$

$\Rightarrow \quad 8 = 4\pi \times r^2 \frac{dr}{dt}$

$\Rightarrow \quad \frac{dr}{dt} = \frac{2}{\pi r^2} \text{ cm/s} \quad \ldots(i)$ (1)

Now, $S = 4\pi r^2$

$\therefore \quad \frac{dS}{dt} = \frac{d}{dt}(4\pi r^2) = 4\pi \times 2r \cdot \frac{dr}{dt}$

$= 8\pi r \times \frac{2}{\pi r^2}$ [using Eq. (i)]

$= \frac{16}{r}$

$\Rightarrow \quad \left(\frac{dS}{dt}\right)_{r=12} = \frac{16}{12} = \frac{4}{3} \text{ cm}^2/\text{s}$ (1)

4. Given, $f(x) = x^3 - 3x^2 + 6x - 100$

On differentiating both sides w.r.t. x, we get

$f'(x) = 3x^2 - 6x + 6 = 3x^2 - 6x + 3 + 3$ (1)

$= 3(x^2 - 2x + 1) + 3 = 3(x-1)^2 + 3 > 0$

$\therefore \quad f'(x) > 0$

This show that function $f(x)$ is increasing on R.

Hence proved. (1)

5. Let r be the radius of sphere and V be its volume.

Then, $V = \frac{4}{3}\pi r^3$

Given, $\frac{dV}{dt} = 3 \text{ cm}^3/\text{s}$

$\therefore \quad \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = 3 \text{ cm}^2/\text{s}$

$\Rightarrow \quad \frac{4}{3}\pi(3r^2)\frac{dr}{dt} = 3$

$\Rightarrow \quad \frac{4}{3}(3\pi r^2)\frac{dr}{dt} = 3$

$\Rightarrow \quad \frac{dr}{dt} = \frac{3}{4\pi r^2} \quad \ldots(i)$ (1)

Now, let S be the surface area of sphere, then

$S = 4\pi r^2$

$\Rightarrow \quad \frac{dS}{dt} = 4\pi(2r)\frac{dr}{dt}$

$\Rightarrow \quad \frac{dS}{dt} = 8\pi r\left(\frac{3}{4\pi r^2}\right)$ [using Eq. (i)]

$\Rightarrow \quad \left(\frac{dS}{dt}\right) = \frac{6}{r}$

when $r = 2$, then

$\frac{dS}{dt} = \frac{6}{2} = 3 \text{ cm}^2/\text{s}$ (1)

6. We have, $f(x) = 4x^3 - 18x^2 + 27x - 7$

On differentiating both sides w.r.t. x, we get

$f'(x) = 12x^2 - 36x + 27$

$\Rightarrow \quad f'(x) = 3(4x^2 - 12x + 9)$

$\Rightarrow \quad f'(x) = 3(2x - 3)^2$ (1)

$\Rightarrow \quad f'(x) \geq 0$

$\Rightarrow$ For any $x \in R, (2x-3)^2 \geq 0$

Since, a perfect square number cannot be negative.

$\therefore$ Given function $f(x)$ is an increasing function on R. (1)

7. Let x be the length of an edge of the cube, V be the volume and S be the surface area at any time t. Then, $V = x^3$ and $S = 6x^2$. (1)

It is given that,

$\frac{dV}{dt} = 8 \text{ cm}^3/\text{sec} \Rightarrow \frac{d}{dt}(x^3) = 8$

$\Rightarrow \quad 3x^2\frac{dx}{dt} = 8 \Rightarrow \frac{dx}{dt} = \frac{8}{3x^2}$ (1)

Now, $S = 6x^2$

$\Rightarrow \quad \frac{dS}{dt} = 12x\frac{dx}{dt} \Rightarrow \frac{dS}{dt} = 12x \times \frac{8}{3x^2}$ (1)

$\Rightarrow \quad \frac{dS}{dt} = \frac{32}{x}$

$\Rightarrow \left(\frac{dS}{dt}\right)_{x=12} = \frac{32}{12} \text{ cm}^2/\text{sec} = \frac{8}{3} \text{ cm}^2/\text{sec}$ (1)

8. We have, $f(x) = \frac{x^4}{4} - x^3 - 5x^2 + 24x + 12$

On differentiating both sides w.r.t. x, we get

$f'(x) = x^3 - 3x^2 - 10x + 24$

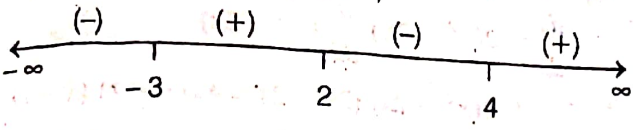
$= (x-2)(x^2 - x - 12)$

$= (x-2)(x^2 - 4x + 3x - 12)$

$= (x-2)(x(x-4) + 3(x-4))$

$= (x-2)(x-4)(x+3)$ (1)

Now, put $f'(x) = 0$, which gives $x = 2, 4$ and -3.
The points $x = -3$, $x = 2$ and $x = 4$ divides the whole real line into four disjoint intervals namely, $(-\infty, -3), (-3, 2), (2, 4), (4, \infty)$



(1)

Note that,

for $x \in (-\infty, -3)$, $f'(x) < 0$

for $x \in (-3, 2)$, $f'(x) > 0$

for $x \in (2, 4)$, $f'(x) < 0$

and for $x \in (4, \infty)$, $f'(x) > 0$ **(1)**

$\therefore f(x)$ is strictly increasing in the intervals $(-3, 2)$ and $(4, \infty)$, and strictly decreasing in the intervals $(-\infty, -3)$ and $(2, 4)$. **(1)**

9. Given, $f(x) = -2x^3 - 9x^2 - 12x + 1$

On differentiating both sides w.r.t. x, we get

$$f'(x) = -6x^2 - 18x - 12 \quad \textbf{(1)}$$

$$\Rightarrow f'(x) = -6(x^2 + 3x + 2)$$

$$\Rightarrow f'(x) = -6(x^2 + 2x + x + 2)$$

$$= -6[x(x + 2) + 1(x + 2)]$$

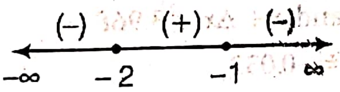
$$= -6(x + 2)(x + 1)$$

Now, put $f'(x) = 0$ **(1)**

$\Rightarrow -6(x + 2)(x + 1) = 0$

$\Rightarrow x = -2, -1$

The points, $x = -2$ and $x = -1$ divide the real line into their disjoint intervals $(-\infty, -2)$, $(-2, -1)$ and $(-1, \infty)$.



The nature of function in these intervals are given below

Interval	Sign of $f'(x)$ $f'(x) = -6(x+2)(x+1)$	Nature of function
$(-\infty, -2)$	$(-)(-)(-) = (-) < 0$	Strictly decreasing
$(-2, -1)$	$(-)(+)(-) = (+) > 0$	Strictly increasing
$(-1, \infty)$	$(-)(+)(+) = (-) < 0$	Strictly decreasing

(1)

Hence, $f(x)$ is strictly increasing in the interval $(-2, -1)$ and $f(x)$ is strictly decreasing in the interval $(-\infty, -2) \cup (-1, \infty)$. **(1)**

10. Using the relation, perimeter of rectangle, $P = 2(x + y)$ and area of rectangle, $A = xy$, differentiate both sides with respect to t and put them in rate of change value and get the result.

Given that length x of a rectangle is decreasing at the rate of 5 cm/min.

$$\therefore \frac{dx}{dt} = -5 \text{ cm/min} \quad \ldots(i)$$

Also, the breadth y of rectangle is increasing at the rate of 4 cm/min.

$$\therefore \frac{dy}{dt} = 4 \text{ cm/min} \quad \ldots(ii) \quad \textbf{(1/2)}$$

(i) Here, we have to find rate of change of perimeter, i.e. dP/dt **(1)**

and we know that, perimeter $P = 2(x + y)$

On differentiating both sides w.r.t. t, we get

$$\frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right)$$

$$\Rightarrow \frac{dP}{dt} = 2(-5 + 4) = 2(-1) = -2 \text{ cm/min}$$

[from Eqs. (i) and (ii)]

Hence, perimeter of rectangle is decreasing at the rate 2 cm/min. **(1)**

(ii) Here, we have to find rate of change of area $\frac{dA}{dt}$.

We know that, area of rectangle $A = xy$

On differentiating both sides w.r.t. t, we get

$$\frac{dA}{dt} = x \cdot \frac{dy}{dt} + y \cdot \frac{dx}{dt}$$

[by using product rule of derivative] **(1/2)**

Now, we have $x = 8$ cm and $y = 6$ cm

$$\frac{dx}{dt} = -5 \text{ cm/min}$$

and $$\frac{dy}{dt} = 4 \text{ cm/min}$$

$$\therefore \frac{dA}{dt} = (8 \times 4) + [6 \times (-5)] = 32 - 30$$

$$\Rightarrow \frac{dA}{dt} = 2 \text{ cm/min}$$

Hence, the area of rectangle is increasing at the rate 2 cm/min. **(1)**

11. Let a be the side of an equilateral triangle and A be the area of an equilateral triangle.

Then, $\frac{da}{dt} = 2$ cm/s

We know that, area of an equilateral triangle,

$$A = \frac{\sqrt{3}}{4}a^2 \quad \textbf{(1)}$$

On differentiating both sides w.r.t. t, we get

$$\frac{dA}{dt} = \frac{\sqrt{3}}{4} \times 2a \times \frac{da}{dt} \quad (1)$$

$$\Rightarrow \frac{dA}{dt} = \frac{\sqrt{3}}{4} \times 2 \times 20 \times 2 \quad [\text{given, } a = 20] \quad (1)$$

$$\therefore \frac{dA}{dt} = 20\sqrt{3} \text{ cm}^2/\text{s}$$

Thus, the rate of area increasing is $20\sqrt{3}$ cm^2/s (1)

12. First, find the first derivative and put equal to zero, we get different values of x and then divide the real line into disjoint intervals. Further check sign of $f(x)$ in a given interval, if $f'(x) > 0$, then it is strictly increasing and if $f'(x) < 0$, then it is strictly decreasing.

Given function is

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$$

On differentiating both sides w.r.t. x, we get

$$f'(x) = 12x^3 - 12x^2 - 24x \quad (1)$$

For strictly increasing or strictly decreasing, put $f'(x) = 0$, we get

$$12x^3 - 12x^2 - 24x = 0$$

$$\Rightarrow 12x(x^2 - x - 2) = 0$$

$$\Rightarrow 12\,x\,[x^2 - 2x + x - 2] = 0$$

$$12\,x\,(x+1)\,(x-2) = 0$$

$$\therefore x = 0, -1 \text{ or } 2 \quad (1)$$

Now, we find intervals in which $f(x)$ is strictly increasing or strictly decreasing.

Interval	$f'(x) = 12x(x+1)(x-2)$	Sign of $f'(x)$
$x < -1$	(−) (−) (−)	− ve
$-1 < x < 0$	(−) (+) (−)	+ ve
$0 < x < 2$	(+) (+) (−)	− ve
$x > 2$	(+) (+) (+)	+ ve

(1)

We know that, a function $f(x)$ is said to be strictly increasing, if $f'(x) > 0$ and it is said to be strictly decreasing, if $f'(x) < 0$. So, the given function $f(x)$ is

(i) strictly increasing on the intervals $(-1, 0)$ and $(2, \infty)$.

(ii) strictly decreasing on the intervals $(-\infty, -1)$ and $(0, 2)$. (1)

13. Do same as Q. No. 12.

Ans. (i) Strictly increasing in $(-2, 1)$ and $(3, \infty)$.

(ii) Strictly decreasing in $(-\infty, -2)$ and $(1, 3)$.

14. Do same as Q. No. 11. [Ans. $10\sqrt{3}$ cm^2/s]

15. Given function is $y = [x(x-2)]^2 = [x^2 - 2x]^2$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 2(x^2 - 2x)\frac{d}{dx}(x^2 - 2x)$$

$$= 2(x^2 - 2x)(2x - 2) = 4x(x-2)(x-1) \quad (1)$$

On putting $\frac{dy}{dx} = 0$, we get

$$4x(x-2)(x-1) = 0 \Rightarrow x = 0, 1 \text{ and } 2 \quad (1)$$

Now, we find interval in which $f(x)$ is strictly increasing or strictly decreasing.

Interval	$\frac{dy}{dx} = 4x(x-2)(x-1)$	Sign of $f'(x)$
$(-\infty, 0)$	(−) (−) (−)	− ve
$(0, 1)$	(+) (−) (−)	+ ve
$(1, 2)$	(+) (−) (+)	− ve
$(2, \infty)$	(+) (+) (+)	+ ve

(1)

Hence, y is strictly increasing in $(0, 1)$ and $(2, \infty)$. Also, y is a polynomial function, so it continuous at $x = 0, 1$ and 2.

Hence, y is increasing in $[0, 1] \cup [2, \infty)$. (1)

16. Let $y = f(x) = (x)^{3/2}$

On differentiating both sides w.r.t. x, we get

$$f'(x) = \frac{3}{2} \cdot x^{1/2} \quad (1)$$

Let $x = 4$ and $x + \Delta x = 3.968$

Then, $\Delta x = -0.032$ (1)

Now, $f(x + \Delta x) \approx f(x) + f'(x)\Delta x$

$$\therefore (x + \Delta x)^{3/2} \approx (x)^{3/2} + \frac{3}{2} \cdot (x)^{1/2} \cdot (-0.032) \quad (1)$$

$$\Rightarrow (4 - 0.032)^{3/2} \approx (4)^{3/2} + \frac{3}{2} \cdot (4)^{1/2} \cdot (-0.032) \quad [\text{put } x = 4]$$

$$\Rightarrow (3.968)^{3/2} \approx 8 + \frac{3}{2} \cdot 2 \cdot (-0.032)$$

$$\Rightarrow (3.968)^{3/2} \approx 8 - 0.096$$

$$\therefore (3.968)^{3/2} \approx 7.904$$

(1)

17. Do same as Q. No. 12.

Ans.

(i) Strictly increasing in $(-3, 0)$ and $(5, \infty)$.

(ii) Strictly decreasing in $(-\infty, -3)$ and $(0, 5)$.

18. First, split 3.02 into two parts x and Δx, so that $x + \Delta x = 3.02$ and $f(x + \Delta x) = f(3.02)$.
Now, write $f(x + \Delta x) \approx f(x) + \Delta x \cdot f'(x)$ and use this result to find the required value.

Given function is $f(x) = 3x^2 + 15x + 3$

On differentiating both sides w.r.t. x, we get

$$f'(x) = 6x + 15$$

Let $x = 3$ and $\Delta x = 0.02$ **(1)**

So that $f(x + \Delta x) = f(3.02)$

By using $f(x + \Delta x) \approx f(x) + \Delta x\, f'(x)$, we get

$f(x + \Delta x) = 3x^2 + 15x + 3 + (6x + 15) \cdot \Delta x$ **(1)**

$f(3 + 0.02) = 3(3)^2 + 15(3) + 3 + [6(3) + 15](0.02)$ **(1)**

$= 27 + 45 + 3 + 33(0.02)$

$= 75 + 0.66$

$= 75.66$

Hence, $f(3.02) \approx 75.66$ **(1)**

19. First, divide 49.5 into two parts as $x = 49$ and $\Delta x = 0.5$. Now, let $y = \sqrt{x}$ and find $\frac{dy}{dx}$. Finally, find Δy using the formula $dy = \frac{dy}{dx} \cdot \Delta x$.

Let $x = 49$, $\Delta x = 0.5$ and $y = \sqrt{x}$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$ **(1)**

At $x = 49$, $\left(\frac{dy}{dx}\right)_{x=49}$

$$= \frac{1}{2\sqrt{49}} = \frac{1}{2 \times 7} = \frac{1}{14}$$ **(1)**

We know that, $dy = \frac{dy}{dx} \cdot \Delta x$

$\Rightarrow$ $dy = \frac{1}{14} \times 0.5$

$= \frac{5}{140} = \frac{1}{28}$ **(1)**

$\therefore$ $\sqrt{49.5} \approx y + dy = \sqrt{49} + \frac{1}{28} = 7 + \frac{1}{28}$

$[\because y = \sqrt{x} = \sqrt{49} = 7]$

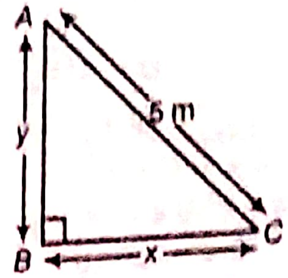
$= \frac{196 + 1}{28} = \frac{197}{28}$ **(1)**

$\therefore$ $\sqrt{49.5} \approx 7.035$

Hence, approximate value of $\sqrt{49.5}$ is 7.035

20. First, draw a rough figure of a right angled triangle, then use Pythagoras theorem. Further differentiate the relation between sides with respect to t and simplify it.

Let AC be the ladder, $BC = x$ and height of the wall, $AB = y$.



As the ladder is pulled along the ground away from the wall at the rate of 2 m/s.

So, $\frac{dx}{dt} = 2$ m/s

To find $\frac{dy}{dt}$, when $x = 4$. **(1)**

In right angled ΔABC, by Pythagoras theorem, we get

$(AB)^2 + (BC)^2 = (AC)^2$

$\Rightarrow$ $x^2 + y^2 = 25$ ...(i)

$\Rightarrow$ $(4)^2 + y^2 = 25 \Rightarrow 16 + y^2 = 25$

$\Rightarrow$ $y^2 = 9$

$\Rightarrow$ $y = \sqrt{9}$

[taking positive square root]

$\therefore$ $y = 3$ **(1)**

On differentiating both sides of Eq. (i) w.r.t. t, we get

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow x\frac{dx}{dt} + y\frac{dy}{dt} = 0$ ...(ii)

[dividing both sides by 2]

On substituting the values of x, y and $\frac{dx}{dt}$ in Eq. (ii), we get

$(4 \times 2) + 3 \times \frac{dy}{dt} = 0$

$\Rightarrow$ $8 + 3 \times \frac{dy}{dt} = 0$ **(1)**

$\therefore$ $\frac{dy}{dt} = \frac{-8}{3}$ m/s

Hence, height of the wall is decreasing at the rate of $\frac{8}{3}$ m/s. **(1)**

NOTE In a rate of change of a quantity, +ve sign shows that it is increasing and −ve sign shows that it is decreasing.

21. Given function is $y = \log(1+x) - \frac{2x}{2+x}$.

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{1+x}(1) - \frac{(2+x)\cdot 2 - 2x\cdot 1}{(2+x)^2}$$ **(1)**

[by using quotient rule of derivative]

$$= \frac{1}{1+x} - \frac{4+2x-2x}{(2+x)^2}$$

$$= \frac{(2+x)^2 - 4(1+x)}{(1+x)(2+x)^2}$$

$$= \frac{4+x^2+4x-4-4x}{(1+x)(2+x)^2}$$

$$= \frac{x^2}{(1+x)(2+x)^2} \quad \text{...(i)}$$ **(1½)**

Now, $x^2, (2+x)^2$ are always positive, also $1+x>0$ for $x>-1$. **(1/2)**

From Eq. (i), $\frac{dy}{dx} > 0$ for $x > -1$.

Hence, function increases for $x > -1$. **(1)**

22. Given function is $f(x) = \sin x + \cos x$.

On differentiating both sides w.r.t. x, we get

$$f'(x) = \cos x - \sin x$$ **(1)**

Now, put $f'(x) = 0 \Rightarrow \cos x - \sin x = 0$

$\Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}$, as $0 \le x \le 2\pi$

Now, we find the intervals in which $f(x)$ is strictly increasing or strictly decreasing. **(1)**

Interval	Test value	$f'(x) = \cos x - \sin x$	Sign of $f'(x)$
$0 < x < \frac{\pi}{4}$	At $x = \frac{\pi}{6}$	$\cos\frac{\pi}{6} - \sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{\sqrt{3}-1}{2}$	+ve
$\frac{\pi}{4} < x < \frac{5\pi}{4}$	At $x = \frac{\pi}{2}$	$\cos\frac{\pi}{2} - \sin\frac{\pi}{2} = 0 - 1 = -1$	−ve
$\frac{5\pi}{4} < x < 2\pi$	At $x = \frac{3\pi}{2}$	$\cos\frac{3\pi}{2} - \sin\frac{3\pi}{2} = 0 - (-1) = 1$	+ve

Note that, $f'(x) > 0$ in $\left(0, \frac{\pi}{4}\right)$, $f'(x) < 0$ in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

and $f'(x) > 0$ in $\left(\frac{5\pi}{4}, 2\pi\right)$. **(1)**

Since, $f(x)$ is a trigonometric function, so it is continuous at $x = 0, \frac{\pi}{4}, \frac{5\pi}{4}$ and 2π.

Hence, the function is

(i) increasing in $\left[0, \frac{\pi}{4}\right]$ and $\left[\frac{5\pi}{4}, 2\pi\right]$

(ii) decreasing in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$ **(1)**

23. Given function is

$$f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$$

On differentiating both sides w.r.t. x, we get

$$f'(x) = 4x^3 - 24x^2 + 44x - 24$$
$$= 4(x^3 - 6x^2 + 11x - 6)$$
$$= 4(x-1)(x^2 - 5x + 6)$$
$$= 4(x-1)(x-2)(x-3)$$ **(1)**

Put $f'(x) = 0$

$\Rightarrow 4(x-1)(x-2)(x-3) = 0 \Rightarrow x = 1, 2, 3$

So, the possible intervals are $(-\infty, 1), (1, 2), (2, 3)$ and $(3, \infty)$. **(1)**

For interval $(-\infty, 1)$, $f'(x) < 0$

For interval $(1, 2)$, $f'(x) > 0$

For interval $(2, 3)$, $f'(x) < 0$

For interval $(3, \infty)$, $f'(x) > 0$.

Also, as $f(x)$ is a polynomial function, so it is continuous at $x = 1, 2, 3$. Hence,

(i) function increases in $[1, 2]$ and $[3, \infty)$. **(1)**

(ii) function decreases in $(-\infty, 1]$ and $[2, 3]$. **(1)**

24. Let V be the volume of cone, h be the height and r be the radius of base of the cone.

Given, $\frac{dV}{dt} = 12\,\text{cm}^3/\text{s}$ ...(i) **(1/2)**

Also, height of cone $= \frac{1}{6} \times$ (radius of base of cone)

$\therefore \quad h = \frac{1}{6}r$ or $r = 6h$ ...(ii) **(1/2)**

We know that, volume of cone is given by

$$V = \frac{1}{3}\pi r^2 h \quad \text{...(iii)}$$

On putting $r = 6h$ from Eq. (ii) in Eq. (iii), we get

$V = \frac{1}{3}\pi(6h)^2 \cdot h \Rightarrow V = \frac{\pi}{3}\cdot 36h^3 \Rightarrow V = 12\pi h^3$ **(1)**

On differentiating both sides w.r.t. t, we get

$$\frac{dV}{dt} = 12\pi \times 3h^2 \cdot \frac{dh}{dt}$$

$\Rightarrow \quad \frac{dV}{dt} = 36\pi h^2 \cdot \frac{dh}{dt}$ **(1)**

On putting $\frac{dV}{dt} = 12\,cm^3/s$ and $h = 4$ cm, we get

$$12 = 36\pi \times 16 \times \frac{dh}{dt}$$

$$\Rightarrow \quad \frac{dh}{dt} = \frac{12}{36\pi \times 16}$$

$$\therefore \quad \frac{dh}{dt} = \frac{1}{48\pi}\ cm/s$$

Hence, the height of sand cone is increasing at the rate of $1/48\pi$ cm/s. **(1)**

25. Let S be the surface area, r be the radius of the sphere.

Given, $r = 9$ cm

Then, dr = Approximate error in radius $r = 0.03$ cm

and dS = Approximate error in surface area **(1)**

Now, we know that surface area of sphere is given by

$$S = 4\pi r^2$$

On differentiating both sides w.r.t. r, we get

$$\frac{dS}{dr} = 4\pi \times 2r = 8\pi r \quad \textbf{(1)}$$

$$\Rightarrow \quad dS = 8\pi r \times dr$$

$$\Rightarrow \quad dS = 8\pi \times 9 \times 0.03$$

$[\because r = 9$ cm and $dr = 0.03$ cm $]$ **(1)**

$$\Rightarrow \quad dS = 72 \times 0.03\pi$$

$$\therefore \quad dS = 2.16\pi\ cm^2/cm$$

Hence, approximate error in surface area is $2.16\pi\ cm^2/cm$. **(1)**

26. Do same as Q. No. 22.

Ans. Strictly increasing in the intervals $\left[0, \frac{\pi}{4}\right)$ and $\left(\frac{5\pi}{4}, 2\pi\right]$ and strictly decreasing in the interval $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$.

27. We know that, a continuous function $y = f(x)$ is said to be increasing on R, if $\frac{dy}{dx} \geq 0, \forall x \in R$. **(1)**

Given, $y = x^3 - 3x^2 + 3x$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 3x^2 - 6x + 3$$

$$\Rightarrow \quad \frac{dy}{dx} = 3(x^2 - 2x + 1)$$

$$\Rightarrow \quad \frac{dy}{dx} = 3(x-1)^2 \quad \textbf{(1)}$$

Now, $3(x-1)^2 \geq 0$ for all real values of x,

i.e. $\forall x \in R$,

$$\frac{dy}{dx} \geq 0, \forall x \in R$$

Hence, the given function is increasing on R. **(2)**

28. Given, $f(x) = (x-1)^3 (x-2)^2$

On differentiating both sides w.r.t. x, we get

$$f'(x) = (x-1)^3 \cdot \frac{d}{dx}(x-2)^2 + (x-2)^2 \cdot \frac{d}{dx}(x-1)^3$$

$$\left[\because \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}\right]$$

$$\Rightarrow f'(x) = (x-1)^3 \cdot 2(x-2) + (x-2)^2 \cdot 3(x-1)^2$$

$$= (x-1)^2 (x-2)[2(x-1) + 3(x-2)]$$

$$= (x-1)^2 (x-2)(2x - 2 + 3x - 6)$$

$$\Rightarrow f'(x) = (x-1)^2 (x-2)(5x-8)$$

Now, put $f'(x) = 0$

$$\Rightarrow \quad (x-1)^2 (x-2)(5x-8) = 0$$

Either $(x-1)^2 = 0$ or $x - 2 = 0$ or $5x - 8 = 0$

$$\therefore \quad x = 1, \frac{8}{5}, 2 \quad \textbf{(1)}$$

Now, we find intervals and check in which interval $f(x)$ is strictly increasing and strictly decreasing.

Interval	$f'(x) = (x-1)^2(x-2)(5x-8)$	Sign of $f'(x)$
$x < 1$	(+)(−)(−)	+ve
$1 < x < \frac{8}{5}$	(+)(−)(−)	+ve
$\frac{8}{5} < x < 2$	(+)(−)(+)	−ve
$x > 2$	(+)(+)(+)	+ve

(1)

We know that, a function $f(x)$ is said to be an strictly increasing function, if $f'(x) > 0$ and strictly decreasing, if $f'(x) < 0$. So, the given function $f(x)$ is increasing on the intervals $(-\infty, 1)$ $\left(1, \frac{8}{5}\right)$ and $(2, \infty)$ and decreasing on $\left(\frac{8}{5}, 2\right)$. **(1)**

Since, $f(x)$ is a polynomial function, so it is continuous at $x = 1, \frac{8}{5}, 2$ Hence, $f(x)$ is

(i) increasing on intervals $\left(-\infty, \frac{8}{5}\right]$ and $[2, \infty)$.

(ii) decreasing on interval $\left[\frac{8}{5}, 2\right]$. **(1)**

NOTE Every strictly increasing (strictly decreasing) function is incresing (decreasing) but converse need not be true.

29. Given function is $f(x) = 2x^3 + 9x^2 + 12x + 20$.

On differentiating both sides w.r.t. x, we get

$$f'(x) = 6x^2 + 18x + 12$$

Put $f'(x) = 0$, we get

$$6x^2 + 18x + 12 = 0$$

$\Rightarrow$ $6(x^2 + 3x + 2) = 0$

$\Rightarrow$ $6(x+1)(x+2) = 0$

$\Rightarrow$ $(x+1)(x+2) = 0$

$\Rightarrow$ $x + 1 = 0$ or $x + 2 = 0$

$\therefore$ $x = -2, -1$ **(1)**

Now, we find intervals and check in which interval $f(x)$ is strictly increasing and strictly decreasing.

Interval	$f'(x) = 6(x+1)(x+2)$	Sign of $f'(x)$
$x < -2$	$(+)(-)(-)$	+ve
$-2 < x < -1$	$(+)(-)(+)$	−ve
$x > -1$	$(+)(+)(+)$	+ve

(1)

We know that, a function $f(x)$ is said to be an strictly increasing function, if $f'(x) > 0$ and strictly decreasing, if $f'(x) < 0$. So, given function is increasing on intervals $(-\infty, -2)$ and $(-1, \infty)$ and decreasing on interval $(-2, -1)$. **(1)**

Since, $f(x)$ is a polynomial function, so it is continuous at $x = -1, -2$.

Hence, given function is

(i) increasing on intervals $(-\infty, -2]$ and $[-1, \infty)$.

(ii) decreasing on interval $[-2, -1]$. **(1)**

30. Do same as Q. No. 29.

Ans. (i) The function increasing on intervals $(-\infty, 1]$ and $[2, \infty)$.

(ii) The function decreasing on interval $[1, 2]$

31. Do same as Q. No. 29.

Ans. The function increasing on $(-\infty, 2]$ and $[3, \infty)$ and decreasing on $[2, 3]$

32. Do same as Q. No. 29.

Ans. (i) The function increasing on $(-\infty, 1]$ and $[2, \infty)$.

(ii) The function decreasing on $[1, 2]$.

33. To prove that given function is increasing, prove that $\frac{dy}{d\theta} \geq 0$ for all θ.

Given function is $y = \frac{4\sin\theta}{2 + \cos\theta} - \theta$ ...(i)

We know that, a function $y = f(x)$ is said to be an increasing function, if $\frac{dy}{dx} \geq 0$, for all values of x. **(1/2)**

On differentiating both sides of Eq. (i) w.r.t. θ, we get

$$\frac{dy}{d\theta} = \frac{\left[(2+\cos\theta) \times \frac{d}{d\theta}(4\sin\theta) - 4\sin\theta \times \frac{d}{d\theta}(2+\cos\theta)\right]}{(2+\cos\theta)^2} - 1$$ **(1)**

[by using quotient rule of derivative]

$$= \frac{(2+\cos\theta)(4\cos\theta) - 4\sin\theta(0 - \sin\theta)}{(2+\cos\theta)^2} - 1$$

$$= \frac{\left[8\cos\theta + 4\cos^2\theta + 4\sin^2\theta - (2+\cos\theta)^2\right]}{(2+\cos\theta)^2}$$

$$= \frac{\left[8\cos\theta + 4(\cos^2\theta + \sin^2\theta) - (4 + \cos^2\theta + 4\cos\theta)\right]}{(2+\cos\theta)^2}$$ **(1)**

$[\because (a+b)^2 = a^2 + b^2 + 2ab]$

$$= \frac{8\cos\theta + 4 - 4 - \cos^2\theta - 4\cos\theta}{(2+\cos\theta)^2}$$

$[\because \sin^2\theta + \cos^2\theta = 1]$

$$= \frac{4\cos\theta - \cos^2\theta}{(2+\cos\theta)^2}$$

$$\Rightarrow \frac{dy}{d\theta} = \frac{\cos\theta(4 - \cos\theta)}{(2+\cos\theta)^2}$$ **(1)**

Now, as $\cos\theta > 0, \forall \theta \in \left(0, \frac{\pi}{2}\right)$

and $(2+\cos\theta)^2$ being a perfect square is always positive for all $\theta \in \left(0, \frac{\pi}{2}\right)$ **(1/2)**

Also, for $\theta \in \left(0, \frac{\pi}{2}\right)$, we know that $0 < \cos\theta < 1$

$\therefore$ $4 - \cos\theta > 0$ for all $\theta \in \left(0, \frac{\pi}{2}\right)$ **(1/2)**

Thus, we conclude that

$$\frac{\cos\theta(4 - \cos\theta)}{(2+\cos\theta)^2} > 0, \forall \theta \in \left(0, \frac{\pi}{2}\right)$$

$$\Rightarrow \frac{dy}{d\theta} > 0, \forall \theta \in \left(0, \frac{\pi}{2}\right)$$ **(1/2)**

$\therefore$ y is an increasing function in $\left(0, \frac{\pi}{2}\right)$. **(1)**

Hence proved.

34. Given, $f(x) = \sin 3x - \cos 3x,\ 0 < x < \pi$

On differentiating both sides w.r.t. x, we get $f'(x) = 3\cos 3x + 3\sin 3x$ **(1)**

On putting $f'(x) = 0$, we get

$\sin 3x = -\cos 3x \Rightarrow \tan 3x = -1 \Rightarrow 3x = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}$ [$\because \tan\theta$ is negative in IInd and IVth quadrants]

$\Rightarrow \quad x = \frac{\pi}{4}, \frac{7\pi}{12}, \frac{11\pi}{12}$ **(1)**

Now, we find intervals and check in which intervals $f(x)$ is strictly increasing or strictly decreasing.

Interval	Test value	$f'(x) = 3(\cos 3x + \sin 3x)$	Sign of $f'(x)$
$0 < x < \frac{\pi}{4}$	At $x = \frac{\pi}{6}$	$3\left(\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right) = 3(0+1) = 3$	+ ve
$\frac{\pi}{4} < x < \frac{7\pi}{12}$	At $x = \frac{\pi}{3}$	$3(\cos\pi + \sin\pi) = 3(-1+0) = -3$	− ve
$\frac{7\pi}{12} < x < \frac{11\pi}{12}$	At $x = \frac{3\pi}{4}$	$3\left(\cos\frac{9\pi}{4} + \sin\frac{9\pi}{4}\right) = 3\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = 3\sqrt{2}$	+ ve
$\frac{11\pi}{12} < x < \pi$	At $x = \frac{23\pi}{24}$	$3\left(\cos\frac{23\pi}{8} + \sin\frac{23}{8}\pi\right)$ $= 3\left[\cos\left(3\pi - \frac{\pi}{8}\right) + \sin\left(3\pi - \frac{\pi}{8}\right)\right] = 3\left(-\cos\frac{\pi}{8} + \sin\frac{\pi}{8}\right)$ $= 3\left(\sin\frac{\pi}{8} - \cos\frac{\pi}{8}\right) < 0$	− ve

(2)

Here, we see that $f'(x) > 0$, for $0 < x < \frac{\pi}{4}$ and $\frac{7\pi}{12} < x < \frac{11\pi}{12}$, so $f(x)$ is strictly increasing in the intervals $\left(0, \frac{\pi}{4}\right)$ and $\left(\frac{7\pi}{12}, \frac{11\pi}{12}\right)$.

While $f'(x) < 0$ in $\frac{\pi}{4} < x < \frac{7\pi}{12}$ and $\frac{11\pi}{12} < x < \pi$ **(1)**

So, $f(x)$ is strictly decreasing in the intervals $\left(\frac{\pi}{4}, \frac{7\pi}{12}\right)$ and $\left(\frac{11\pi}{12}, \pi\right)$. **(1)**

35. Given function is $f(x) = x^2 - x + 1$.

On differentiating both sides w.r.t. x, we get $f'(x) = 2x - 1$ **(1)**

On putting $f'(x) = 0 \Rightarrow 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$ **(1)**

Now, we find intervals in which $f(x)$ is strictly increasing or strictly decreasing.

Interval	$f'(x) = (2x-1)$	Sign of $f'(x)$
$x < \frac{1}{2}$	(−)	− ve
$x > \frac{1}{2}$	(+)	+ ve

(1)

Here, $f(x)$ is strictly increasing on $\left(\frac{1}{2}, \infty\right)$ and $f(x)$ is strictly decreasing on $\left(-\infty, \frac{1}{2}\right)$. **(1)**

$\Rightarrow f(x)$ is strictly increasing on $\left(\frac{1}{2}, 1\right)$ and $f(x)$ is strictly decreasing on $\left(-1, \frac{1}{2}\right)$ **(1)**

$\therefore f'(x)$ does not have same sign throughout the interval $(-1, 1)$.

Thus, $f(x)$ is neither increasing nor decreasing in $(-1,1)$. **(1)**

36. Given function is $f(x) = 20 - 9x + 6x^2 - x^3$. **(1)**

On differentiating both sides w.r.t. x, we get $f'(x) = -9 + 12x - 3x^2$

On putting $f'(x) = 0$, we get

$$-9 + 12x - 3x^2 = 0 \Rightarrow -3(x^2 - 4x + 3) = 0 \Rightarrow -3(x-1)(x-3) = 0 \Rightarrow (x-1)(x-3) = 0$$ **(2)**

$$\Rightarrow \quad x - 1 = 0 \text{ or } x - 3 = 0 \quad \Rightarrow x = 1 \text{ or } 3$$

Now, we find intervals in which $f(x)$ is strictly increasing or strictly decreasing.

Interval	$f'(x) = -3(x-1)(x-3)$	Sign of $f'(x)$
$x < 1$	$(-)(-)(-)$	– ve
$1 < x < 3$	$(-)(+)(-)$	+ ve
$x > 3$	$(-)(+)(+)$	– ve

(1)

We know that, a function $f(x)$ is said to be strictly increasing when $f'(x) > 0$ and it is said to be strictly decreasing, if $f'(x) < 0$. So, the given function $f(x)$ is

(i) strictly increasing on the interval $(1, 3)$ and

(ii) strictly decreasing on the intervals $(-\infty, 1)$ and $(3, \infty)$. **(2)**

37. Given function is $f(x) = \sin x - \cos x, 0 \le x \le 2\pi$ **(1)**

On differentiating both sides w.r.t. x, we get $f'(x) = \cos x + \sin x$

On putting $f'(x) = 0$, we get

$$\cos x + \sin x = 0 \Rightarrow \sin x = -\cos x \Rightarrow \frac{\sin x}{\cos x} = -1 \Rightarrow \tan x = -1$$

For $\quad x \in [0, 2\pi]$, $\tan x = \tan\frac{3\pi}{4} \Rightarrow x = \frac{3\pi}{4}$

or $\quad \tan x = \tan\frac{7\pi}{4} \Rightarrow x = \frac{7\pi}{4}$

$x = \frac{3\pi}{4}, \frac{7\pi}{4}$ $\quad$ [$\because \tan\theta$ is negative in IInd quadrant and IVth quadrant] **(2)**

Now, we find the intervals in which $f(x)$ is strictly increasing or strictly decreasing.

Interval	Test value	$f'(x) = \cos x + \sin x$	Sign of $f'(x)$
$0 < x < \frac{3\pi}{4}$	At $x = \frac{\pi}{2}$	$f'\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} + \sin\frac{\pi}{2} = 0 + 1 = 1$	+ ve
$\frac{3\pi}{4} < x < \frac{7\pi}{4}$	At $x = \frac{5\pi}{6}$	$f'\left(\frac{5\pi}{6}\right) = \cos\frac{5\pi}{6} + \sin\frac{5\pi}{6}$ $= \cos\left(\pi - \frac{\pi}{6}\right) + \sin\left(\pi - \frac{\pi}{6}\right)$ $= -\cos\frac{\pi}{6} + \sin\frac{\pi}{6} = \frac{-\sqrt{3}}{2} + \frac{1}{2} = \frac{-\sqrt{3}+1}{2}$	– ve
$\frac{7\pi}{4} < x < 2\pi$	At $x = \frac{23\pi}{12}$	$f'\left(\frac{23\pi}{12}\right) = \cos\frac{23\pi}{12} + \sin\frac{23\pi}{12}$ $= \cos\left(2\pi - \frac{\pi}{12}\right) + \sin\left(2\pi - \frac{\pi}{12}\right) = \cos\frac{\pi}{12} - \sin\frac{\pi}{12} > 0$	+ ve

(2)

We know that, a function $f(x)$ is said to be strictly increasing in an interval when $f'(x) > 0$ and it is said to be strictly decreasing when $f'(x) < 0$. So, the given function $f(x)$ is strictly increasing in intervals $\left(0, \frac{3\pi}{4}\right)$ and $\left(\frac{7\pi}{4}, 2\pi\right)$ and it is strictly decreasing in the interval $\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$.

(1)

Solutions

1. Let (x_1, y_1) be the point on curve from which tangent to be taken. We know that, slope of tangent is given by

$$\left.\frac{dy}{dx}\right|_{\text{at point of contact}}$$

Here, $y = \sqrt{3x-2}$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{3}{2\sqrt{3x-2}}.$$

Slope of tangent at (x_1, y_1) is

$$\left.\frac{dy}{dx}\right|_{(x_1, y_1)} = \frac{3}{2\sqrt{3x_1-2}}$$ **(1)**

Given, tangent is parallel to the line $4x - 2y + 5 = 0$

So, slope of tangent = slope of $(4x - 2y + 5 = 0)$

$$= \frac{-4}{-2} = 2$$

$$\left[\because \text{slope of } ax + by + c = 0 \text{ is } -\frac{a}{b}\right]$$

$$\Rightarrow \quad \frac{3}{2\sqrt{3x_1-2}} = 2$$

$$\Rightarrow \quad 3 = 4\sqrt{3x_1-2}$$

On squaring both sides, we get

$$9 = 16(3x_1 - 2)$$

$$\Rightarrow \quad 3x_1 = \frac{9}{16} + 2 \Rightarrow x_1 = \frac{41}{48}$$ **(1)**

Since, point (x_1, y_1) is on the curve, therefore $y_1 = \sqrt{3x_1 - 2}$

$$\Rightarrow \quad y_1 = \sqrt{3 \times \frac{41}{48} - 2} = \frac{3}{4}$$

Hence, the point is $(x_1, y_1) = \left(\frac{41}{48}, \frac{3}{4}\right)$

Now, equation of the tangent is given by

$$y - y_1 = m(x - x_1)$$

$$\Rightarrow \quad y - \frac{3}{4} = 2\left(x - \frac{41}{48}\right)$$

$$\Rightarrow \quad \frac{4y-3}{4} = \frac{48x-41}{24}$$

$$\Rightarrow \quad \frac{24}{4}(4y-3) = 48x - 41$$

$$\Rightarrow \quad 6(4y-3) = 48x - 41$$

$$\Rightarrow \quad 48x - 41 - 24y + 18 = 0$$

$$\Rightarrow \quad 48x - 24y = 23$$

which is the required equation of the tangent. **(1)**

Now, equation of the normal is given by

$$y - y_1 = \frac{-1}{m}(x - x_1)$$

$$\Rightarrow \quad y - \frac{3}{4} = \frac{-1}{2}\left(x - \frac{41}{48}\right)$$

$$\Rightarrow \quad \frac{4y-3}{4} = \frac{-1}{2}\left(\frac{48x-41}{48}\right)$$

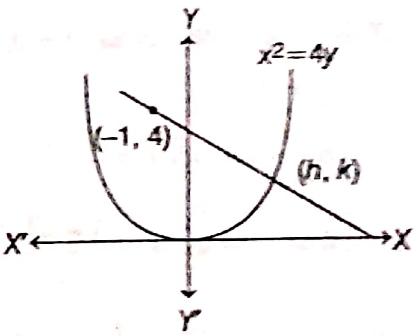
$$\Rightarrow \quad 4y - 3 = \frac{-48x+41}{24}$$

$$\Rightarrow \quad 24(4y-3) = 41 - 48x$$

$$\Rightarrow \quad 96y + 48x - 113 = 0,$$

which is the required equation of the normal. **(1)**

2. Given curve is $x^2 = 4y$



On differentiating w.r.t. x, we get

$$2x = 4\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2x}{4} = \frac{x}{2}$$

$$\text{Slope of normal} = -\frac{1}{\frac{dy}{dx}} = -\frac{1}{\frac{x}{2}} = -\frac{2}{x}$$ **(1)**

Let (h, k) be the point where normal and curve intersect.

Then, slope of normal at $(h, k) = -\frac{2}{h}$

Equation of normal passing through (h, k) with slope $\frac{-2}{h}$ is

$$y - k = \frac{-2}{h}(x - h) \quad \text{(1)}$$

Since normal passes through $(-1, 4)$. It will satisfy its equation

$$\Rightarrow \quad 4 - k = \frac{-2}{h}(-1 - h)$$

$$\Rightarrow \quad k = 4 + \frac{2}{h}(-1 - h) \quad \text{...(i)}$$

Since, (h, k) lies on curve $x^2 = 4y$

$$\therefore \quad h^2 = 4k \Rightarrow k = \frac{h^2}{4} \quad \text{...(ii)}$$

Using Eqs. (i) and (ii), we get

$$4 + \frac{2}{h}(-1 - h) = \frac{h^2}{4}$$

$$\Rightarrow \quad 4 - \frac{2}{h}(1 + h) = \frac{h^2}{4}$$

$$\Rightarrow \quad 4 - \frac{2}{h} - 2 = \frac{h^2}{4}$$

$$\Rightarrow \quad 2 - \frac{2}{h} = \frac{h^2}{4}$$

$$\Rightarrow \quad 2h - 2 = \frac{h^3}{4}$$

$$\Rightarrow \quad h^3 - 8h + 8 = 0$$

$$\Rightarrow \quad (h - 2)(h^2 + 2h - 4) = 0 \Rightarrow h = 2, -1 \pm \sqrt{5}$$

When, $h = 2, k = 1$

$$h = -1 + \sqrt{5}, k = \frac{3 - \sqrt{5}}{2}$$

$$h = -1 - \sqrt{5}, k = \frac{3 + \sqrt{5}}{2} \quad \text{(1)}$$

$\therefore$ Required points are $(2, 1)$, $\left(-1 + \sqrt{5}, \frac{3 - \sqrt{5}}{2}\right)$ and $\left(-1 - \sqrt{5}, \frac{3 + \sqrt{5}}{2}\right)$.

Therefore, equation of normal at point $(2, 1)$ are

$$y - 1 = -\frac{2}{2}(x - 2)$$

$$\Rightarrow \quad y - 1 = -x + 2$$

$$\Rightarrow \quad x + y = 3$$

Similarly at points $\left(-1 + \sqrt{5}, \frac{3 - \sqrt{5}}{2}\right)$ and $\left(-1 - \sqrt{5}, \frac{3 + \sqrt{5}}{2}\right)$ are

$$y - \left(\frac{3 - \sqrt{5}}{2}\right) = \frac{-2}{\sqrt{5} - 1}\{x - (\sqrt{5} - 1)\}$$

and $$y - \left(\frac{3 + \sqrt{5}}{2}\right) = \frac{-2}{-1 - \sqrt{5}}\{x - (-1 - \sqrt{5})\} \quad \text{(1)}$$

3. Given equation of curve is

$$16x^2 + 9y^2 = 145 \quad \text{...(i)}$$

Clearly, when $x = 2$, then

$$16(2)^2 + 9y^2 = 145$$

$$\Rightarrow \quad 9y^2 = 145 - 64$$

$$\Rightarrow \quad 9y^2 = 81$$

$$\Rightarrow \quad y^2 = 9$$

$$\Rightarrow \quad y = \pm 3$$

[taking square root on both sides]

But it is given that $y_1 > 0$

$$\therefore \quad y = 3 \quad \text{(1/2)}$$

Thus, the point of contact $(x_1, y_1) = (2, 3)$

Now, on differentiating both sides of Eq. (i) w.r.t. x, we get

$$32x + 18y\frac{dy}{dx} = 0 \Rightarrow 18y\frac{dy}{dx} = -32x$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{16}{9} \cdot \frac{x}{y} \quad \text{(1/2)}$$

$\therefore$ Slope of tangent at point $(2, 3)$

$$= \left(\frac{dy}{dx}\right)_{(2,3)} = \frac{-16}{9} \times \frac{2}{3} = \frac{-32}{27} \quad \text{(1/2)}$$

and slope of normal at point $(2, 3)$

$$= \frac{-1}{\left(\frac{dy}{dx}\right)_{(2,3)}} = \frac{-1}{\left(\frac{-32}{27}\right)} = \frac{27}{32} \quad \text{(1/2)}$$

Now, equation of tangent at point $(2, 3)$ is

$$(y - 3) = \frac{-32}{27}(x - 2)$$

$$\Rightarrow \quad 27y - 81 = -32x + 64 \Rightarrow 32x + 27y = 145 \quad \text{(1)}$$

and equation of normal at point $(2, 3)$ is

$$(y - 3) = \frac{27}{32}(x - 2)$$

$$\Rightarrow \quad 32y - 96 = 27x - 54$$

$$\Rightarrow \quad 27x - 32y = -42 \quad \text{(1)}$$

4. Given curves are $x^2 + y^2 = 4$ ...(i)

and $(x-2)^2 + y^2 = 4$ ...(ii)

From Eqs. (i) and (ii), we get

$$x^2 + 4 - 4x + y^2 = 4$$

$$\Rightarrow \quad 4 - 4x = 0 \Rightarrow x = 1 \quad \textbf{(1)}$$

On putting value of x in Eq. (i), we get

$$y = \sqrt{3}$$

[since point taken in first quadrant]

$\therefore$ Point of intersection is $(1, \sqrt{3})$.

On differentiating both sides of Eq. (i), we get

$$2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-x}{y}$$

Then, $m_1 = \left(\frac{dy}{dx}\right)_{(1,\sqrt{3})} = \frac{-1}{\sqrt{3}}$ **(1)**

On differentiating both sides of Eq. (ii), we get

$$2(x-2) + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-(x-2)}{y}$$

Then, $m_2 = \left(\frac{dy}{dx}\right)_{(1,\sqrt{3})} = \frac{-(1-2)}{\sqrt{3}} = \frac{1}{\sqrt{3}}$ **(1)**

If θ is the angle between both curves (i) and (ii), then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{\frac{-1}{\sqrt{3}} - \frac{1}{\sqrt{3}}}{1 + \left(\frac{-1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right)}\right| = \left|\frac{\frac{-2}{\sqrt{3}}}{1 - \frac{1}{3}}\right|$$

$$= \frac{2}{\sqrt{3}} \times \frac{3}{2} = \sqrt{3}$$

$$\therefore \theta = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$$

$\therefore$ Angle of intersection between the curves is $\frac{\pi}{3}$. **(1)**

5. Given equations of curves are

$$x = 3\cos t - \cos^3 t$$

and $y = 3\sin t - \sin^3 t$

On differentiating w.r.t. t, we get

$$\frac{dx}{dt} = 3\frac{d}{dt}(\cos t) - \frac{d}{dt}(\cos^3 t)$$

$$\Rightarrow \quad \frac{dx}{dt} = 3 \times (-\sin t) - 3\cos^2 t \frac{d}{dt}(\cos t)$$

$$\Rightarrow \quad \frac{dx}{dt} = -3\sin t - 3\cos^2 t \times (-\sin t)$$

$$\Rightarrow \quad \frac{dx}{dt} = -3\sin t(1 - \cos^2 t)$$

$$\Rightarrow \quad \frac{dx}{dt} = -3\sin t \times \sin^2 t \quad \textbf{(1)}$$

and $\frac{dy}{dt} = 3\frac{d}{dt}(\sin t) - \frac{d}{dt}(\sin^3 t)$

$$\Rightarrow \quad \frac{dy}{dt} = 3 \times (\cos t) - 3\sin^2 t \frac{d}{dt}(\sin t)$$

$$\Rightarrow \quad \frac{dy}{dt} = 3\cos t - 3\sin^2 t \times \cos t$$

$$\Rightarrow \quad \frac{dy}{dt} = 3\cos t(1 - \sin^2 t)$$

$$\Rightarrow \quad \frac{dy}{dt} = 3\cos t \times \cos^2 t \quad \textbf{(1)}$$

$$\therefore \quad \frac{dx}{dt} = -3\sin^3 t \text{ and } \frac{dy}{dt} = 3\cos^3 t$$

Now, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3\cos^3 t}{-3\sin^3 t} = -\cot^3 t$ **(1/2)**

$\therefore$ Equation of normal is

$$y - (3\sin t - \sin^3 t) = -\frac{1}{-\cot^3 t}[x - (3\cos t - \cos^3 t)] \quad \textbf{(1/2)}$$

$$\left[\because y - y_1 = -\frac{1}{dy/dx}(x - x_1)\right]$$

$$\Rightarrow (y - 3\sin t + \sin^3 t) = \frac{\sin^3 t}{\cos^3 t}(x - 3\cos t + \cos^3 t)$$

$$\Rightarrow \quad y\cos^3 t - 3\sin t\cos^3 t + \sin^3 t\cos^3 t$$

$$= x\sin^3 t - 3\sin^3 t\cos t + \sin^3 t\cos^3 t \quad \textbf{(1/2)}$$

$$\Rightarrow y\cos^3 t - x\sin^3 t = 3\sin t\cos^3 t - 3\sin^3 t\cos t$$

$$\Rightarrow y\cos^3 t - x\sin^3 t = 3\sin t\cos t(\cos^2 t - \sin^2 t)$$

$$= 3\sin t\cos t(\cos 2t) \times \frac{2}{2}$$

$$[\because \cos^2 t - \sin^2 t = \cos 2t]$$

$$= \frac{3}{2}\sin 2t \times \cos 2t \quad [\because 2\sin t\cos t = \sin 2t]$$

$$= \frac{3}{2}\sin 2t \times \cos 2t \times \frac{2}{2}$$

$$= \frac{3}{4}\sin 4t \quad [\because 2\sin 2t\cos 2t = \sin 4t]$$

$$\therefore 4(y\cos^3 t - x\sin^3 t) = 3\sin 4t$$

(1/2)

Hence proved.

6. Given equation of curve is

$$y = x^3 + 2x - 4$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 3x^2 + 2$$

$\therefore$ The slope of required tangent is

$$m_1 = \frac{dy}{dx} = 3x^2 + 2 \quad \textbf{(1/2)}$$

Now, slope of line $x + 14y - 3 = 0$

or $$y = -\frac{x}{14} + \frac{3}{14}$$

is $$m_2 = -\frac{1}{14} \quad \textbf{(1/2)}$$

Since, the required tangent is perpendicular to the line $x + 14y - 3 = 0$.

$$\therefore \quad m_1 m_2 = -1$$

$$\Rightarrow \quad (3x^2 + 2) \times \left(-\frac{1}{14}\right) = -1$$

$$\Rightarrow \quad 3x^2 + 2 = 14$$

$$\Rightarrow \quad 3x^2 = 12 \Rightarrow x^2 = 4$$

$$\Rightarrow \quad x = \pm 2 \quad \textbf{(1)}$$

When $x = 2$, then

$$y = 2^3 + 2 \times 2 - 4 = 8 + 4 - 4 = 8$$

When $x = -2$, then

$$y = (-2)^3 + 2 \times (-2) - 4$$
$$= -8 - 4 - 4 = -16$$

$\therefore$ Points of contact are $(2, 8)$ and $(-2, -16)$. **(1/2)**

Now, equation of tangent at point $(2, 8)$ is

$$y - 8 = \left(\frac{dy}{dx}\right)_{(2,8)} (x - 2)$$

$$\Rightarrow \quad y - 8 = (3 \times 2^2 + 2)(x - 2)$$

$$\Rightarrow \quad y - 8 = 14(x - 2)$$

$$\Rightarrow \quad y = 14x - 20 \quad \textbf{(1/2)}$$

and equation of tangent at point $(-2, -16)$ is

$$y + 16 = \frac{dy}{dx}_{(-2,-16)} (x + 2)$$

$$\Rightarrow \quad y + 16 = [3(-2)^2 + 2](x + 2)$$

$$\Rightarrow \quad y + 16 = 14(x + 2)$$

$$\therefore \quad y = 14x + 12$$

Hence, the required equation of tangents are $y = 14x - 20$ and $y = 14x + 12$ **(1)**

7. Given equation of curve is $y^2 = ax^3 + b$ ...(i)

and equation of tangent is $y = 4x - 5$ ...(ii)

On differentiating both sides of Eq. (i) w.r.t. x, we get

$$2y\frac{dy}{dx} = 3ax^2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3ax^2}{2y}$$

Now, slope of tangent at $(2,3) = \left(\frac{dy}{dx}\right)_{(2,3)}$

$$= \frac{3a(2)^2}{2(3)} = 2a \quad \textbf{(1)}$$

But, from Eq. (ii), we have slope of tangent = 4

$$\therefore \quad 2a = 4 \Rightarrow a = 2 \quad \textbf{(1)}$$

Since, $(2, 3)$ lies on the curve, therefore from Eq. (i), we get

$$9 = 8a + b \quad \textbf{(1)}$$

$$\Rightarrow \quad 9 = 16 + b \quad [\because a = 2]$$

$$\Rightarrow \quad b = -7$$

Hence, $a = 2$ and $b = -7$ **(1)**

8. Given curve is $9y^2 = x^3$. ...(i)

On differentiating both sides w.r.t. x, we get

$$9 \times 2y\frac{dy}{dx} = 3x^2 \Rightarrow \frac{dy}{dx} = \frac{3x^2}{18y} = \frac{x^2}{6y}$$

Let (x_1, y_1) be the required point on the curve (i). Then, slope of normal to the curve (i) at point (x_1, y_1)

$$= -\frac{1}{(dy/dx)_{(x_1, y_1)}} = \frac{-6y_1}{x_1^2} \quad \text{...(ii)} \quad \textbf{(1)}$$

Since, normal makes equal intercept on the axes.

$\therefore$ Slope of normal $= \tan\frac{\pi}{4}, \tan\frac{3\pi}{4} = \pm 1$ **(1)**

Hence, $$\frac{-6y_1}{x_1^2} = \pm 1 \quad \text{[from Eq. (ii)]}$$

$$\Rightarrow \quad y_1 = \pm\frac{x_1^2}{6} \quad \text{...(iii)}$$

Again, point (x_1, y_1) lies on curve (i).

$$\therefore \quad 9y_1^2 = x_1^3$$

$$\Rightarrow \quad 9\left(\pm\frac{x_1^2}{6}\right)^2 = x_1^3 \quad \text{[using Eq. (iii)]}$$

$$\Rightarrow \quad \frac{9x_1^4}{36} = x_1^3 \quad \textbf{(1)}$$

$$\Rightarrow \quad \frac{x_1}{4} = 1$$

$$\Rightarrow \quad x_1 = 4$$

$$\therefore \quad y_1 = \pm\frac{x_1^2}{6} = \pm\frac{16}{6} = \pm\frac{8}{3}$$

Hence, required points are $\left(4, \frac{8}{3}\right)$ and $\left(4, \frac{-8}{3}\right)$. **(1)**

Note that $x_1 \neq 0$, as $x_1 = 0$, then $y_1 = 0$. So, normal will pass through $(0, 0)$, which is not possible.

9. Given, $x = a \sin^3\theta$

On differentiating both sides w.r.t. θ, we get

$$\frac{dx}{d\theta} = a(3\sin^2\theta \cos\theta)$$

$$\Rightarrow \frac{dx}{d\theta} = 3a \sin^2\theta \cos\theta$$ **(1/2)**

and $y = a \cos^3\theta$

On differentiating both sides w.r.t. θ, we get

$$\frac{dy}{d\theta} = -3a \cos^2\theta \sin\theta$$

Then, $$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-3a\cos^2\theta \sin\theta}{3a\sin^2\theta\cos\theta}$$

$$\Rightarrow \frac{dy}{dx} = -\cot\theta$$ **(1/2)**

$$\therefore \left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{4}} = -\cot\frac{\pi}{4} = -1$$

Also, at $\theta = \frac{\pi}{4}$,

$$x = a\left(\sin\frac{\pi}{4}\right)^3, y = a\left(\cos\frac{\pi}{4}\right)^3$$

$$\Rightarrow x = a\left(\frac{1}{2}\right)^{3/2}, y = a\left(\frac{1}{2}\right)^{3/2}$$ **(1)**

$$\left[\because \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \text{ and } \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}\right]$$

Now, equation of tangent at the point $\left(\frac{a}{(2)^{3/2}}, \frac{a}{(2)^{3/2}}\right)$ is

$$y - y_1 = \frac{dy}{dx}(x - x_1)$$

$$\Rightarrow y - \frac{a}{(2)^{3/2}} = (-1)\left(x - \frac{a}{2^{3/2}}\right)$$

$$\Rightarrow y + x = \frac{2a}{(2)^{3/2}} \Rightarrow y + x = \frac{a}{2\sqrt{2}}$$

$$\Rightarrow x + y - \frac{a}{2\sqrt{2}} = 0$$ **(1)**

Clearly, slope of normal $= \frac{-1}{\text{Slope of tangent}}$

$\Rightarrow$ Slope of normal $= \frac{-1}{-1} = 1$

Now, equation of normal at the point $\left(\frac{a}{(2)^{3/2}}, \frac{a}{(2)^{3/2}}\right)$ is $y - \frac{a}{(2)^{3/2}} = (1)\left(x - \frac{a}{2^{3/2}}\right)$

$$\therefore x - y = 0$$ **(1)**

10. Given equation of curve is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

On differentiating both sides w.r.t. x, we get

$$\frac{2x}{a^2} - \frac{2y}{b^2}\cdot\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{b^2x}{a^2y}$$ **(1)**

$\therefore$ Slope of the tangent at point $(\sqrt{2}a, b)$ is

$$\left(\frac{dy}{dx}\right)_{(\sqrt{2}a, b)} = \frac{\sqrt{2}ab^2}{ba^2} = \frac{\sqrt{2}b}{a}$$ **(1/2)**

Hence, the equation of the tangent at point $(\sqrt{2}a, b)$ is

$$y - b = \frac{\sqrt{2}b}{a}(x - \sqrt{2}a)$$

$$\Rightarrow a(y - b) = \sqrt{2}b(x - \sqrt{2}a)$$

$$\Rightarrow ay - ab = \sqrt{2}bx - 2ab$$

$$\Rightarrow \sqrt{2}bx - ay - ab = 0$$ **(1)**

Now, the slope of the normal at point $(\sqrt{2}a, b)$

$$= \frac{-1}{\text{Slope of tangent}} = \frac{-1}{\sqrt{2}b/a}$$ **(1/2)**

Hence, the equation of the normal at point$(\sqrt{2}a, b)$ is

$$(y - b) = -\frac{a}{\sqrt{2}b}(x - \sqrt{2}a)$$

$$\Rightarrow \sqrt{2}b(y - b) = -a(x - \sqrt{2}a)$$

$$\Rightarrow \sqrt{2}by - \sqrt{2}b^2 = -ax + \sqrt{2}a^2$$

$$\therefore ax + \sqrt{2}by - \sqrt{2}(a^2 + b^2) = 0$$ **(1)**

11. First, find the slope of tangent to the given curve and of given equation of tangent, then equate them to get value of x. Put value of x in given curve to find required points.

Given, equation of curve is

$$y = x^3 - 11x + 5 \quad \text{...(i)}$$

Slope of the tangent at any point (x, y) is given by $\frac{dy}{dx}$.

$$\therefore \frac{dy}{dx} = 3x^2 - 11 \quad \text{...(ii)}$$ **(1)**

Also, slope of the tangent $y = x - 11$ is 1.

$$\therefore \frac{dy}{dx} = 1$$

$$\Rightarrow 3x^2 - 11 = 1 \quad \text{[from Eq. (i)]}$$

$$\Rightarrow 3x^2 = 12 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$$ **(1)**

When $x = 2$, then $y = (2)^3 - 11(2) + 5$

$$= 8 - 22 + 5 = -9 \quad \textbf{(1/2)}$$

When $x = -2$, then $y = (-2)^3 - 11(-2) + 5$

$$= -8 + 22 + 5 = 19 \quad \textbf{(1/2)}$$

Since, the points (–2; 19) does not lies on the line $y = x - 1$

Hence, the required points on the curve are (2, –9). **(1)**

12. As tangent is parallel to X-axis, put $\frac{dy}{dx} = 0$ and find value of x from it. Then, put this value of x in the equation of the given curve and find value of y.

Given equation of curve is

$$x^2 + y^2 - 2x - 3 = 0 \quad \text{...(i)}$$

On differentiating both sides of Eq. (i) w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} - 2 = 0$$

$$\Rightarrow \quad 2y\frac{dy}{dx} = 2 - 2x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2-2x}{2y} \Rightarrow \frac{dy}{dx} = \frac{1-x}{y} \quad \textbf{(1)}$$

We know that, when a tangent to the curve is parallel to X-axis, then $\frac{dy}{dx} = 0$. **(1)**

On putting $\frac{dy}{dx} = 0$, we get

$$1 - x = 0 \Rightarrow x = 1 \quad \textbf{(1/2)}$$

Now, on putting $x = 1$ in Eq. (i), we get

$$1 + y^2 - 2 - 3 = 0$$

$$\Rightarrow \quad y^2 - 4 = 0$$

$$\Rightarrow \quad y^2 = 4 \Rightarrow y = \pm 2 \quad \textbf{(1/2)}$$

Hence, the required points are (1, 2) and (1, –2). **(1)**

13. First, determine the derivative and put $\frac{dy}{dx} = y$ and then find the value of x from it. Further, put this value of x in the equation of the given curve and find the value of y.

Given equation of curve is $y = x^3$. ...(i)

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 3x^2$$

$\therefore$ Slope of tangent at any point (x, y) is

$$\frac{dy}{dx} = 3x^2 \quad \textbf{(1)}$$

Now, given that

Slope of tangent = y-coordinate of the point

$$\Rightarrow \quad \frac{dy}{dx} = y$$

$$\Rightarrow \quad 3x^2 = y \quad \left[\because \frac{dy}{dx} = 3x^2\right]$$

$$\Rightarrow \quad 3x^2 = x^3 \quad [\because y = x^3]$$

$$\Rightarrow \quad 3x^2 - x^3 = 0 \Rightarrow x^2(3 - x) = 0$$

$\Rightarrow$ Either $x^2 = 0$ or $3 - x = 0$

$$\therefore \quad x = 0, 3 \quad \textbf{(1)}$$

Now, on putting $x = 0$ and 3 in Eq. (i), we get

$$y = (0)^3 = 0 \quad [\text{at } x = 0]$$

and $$y = (3)^3 = 27 \quad [\text{at } x = 3] \quad \textbf{(1)}$$

Hence, the required points are (0, 0) and (3, 27). **(1)**

14. We know that, the equation of tangent at the point (x_1, y_1) is $y - y_1 = m(x - x_1)$ ...(i)

where, $$m = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} \quad \textbf{(1/2)}$$

Now, given $x = \sin 3t$ ...(ii)

$$\therefore \quad \frac{dx}{dt} = 3\cos 3t \quad [\text{differentiate w.r.t. } t]$$

and $y = \cos 2t$ ...(iii)

$$\therefore \quad \frac{dy}{dt} = -2\sin 2t \quad [\text{differentiate w.r.t. } t]$$

Then, $$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{-2\sin 2t}{3\cos 3t} \quad \textbf{(1)}$$

On putting $t = \frac{\pi}{4}$, we get

$$m = \left(\frac{dy}{dx}\right)_{t=\frac{\pi}{4}} = \frac{-2\sin\frac{\pi}{2}}{3\cos\frac{3\pi}{4}} = \frac{-2}{-\frac{3}{\sqrt{2}}}$$

$$\left[\because \sin\frac{\pi}{2} = 1 \text{ and } \cos\frac{3\pi}{4} = \cos\left(\pi - \frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}\right]$$

$$\Rightarrow \quad m = \frac{2\sqrt{2}}{3} \quad \textbf{(1)}$$

Also, to find (x_1, y_1), we put $t = \frac{\pi}{4}$ in given curves.

From Eqs. (ii) and (iii), we get

$$x_1 = \sin\frac{3\pi}{4} = \sin\left(\pi - \frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

and $y_1 = \cos\frac{\pi}{2} = 0$

$$\therefore \quad (x_1, y_1) = \left(\frac{1}{\sqrt{2}}, 0\right) \quad \textbf{(1/2)}$$

Now, on putting $(x_1, y_1) = \left(\frac{1}{\sqrt{2}}, 0\right)$ and $m = \frac{2\sqrt{2}}{3}$ in Eq. (i), we get

$$y - 0 = \frac{2\sqrt{2}}{3}\left(x - \frac{1}{\sqrt{2}}\right) \Rightarrow y - 0 = \frac{2\sqrt{2}x}{3} - \frac{2}{3}$$

$$\Rightarrow 3y = 2\sqrt{2}x - 2$$

Hence, required equation of tangent is

$$2\sqrt{2}x - 3y - 2 = 0. \quad \textbf{(1)}$$

15. The curve cuts the X-axis, so put $y = 0$ and get the corresponding values of x. Further, differentiate and determine the slopes at different points. And then use the equation of tangent

$$y - y_1 = m(x - x_1).$$

Given equation of the curve is

$$y = (x^2 - 1)(x - 2) \quad \text{...(i)}$$

Since, the curve cuts the X-axis, so at that point y- coordinate will be zero.

So, on putting $y = 0$, we get

$$(x^2 - 1)(x - 2) = 0 \Rightarrow x^2 = 1 \text{ or } x = 2$$

$$\therefore \quad x = \pm 1 \text{ or } 2 \Rightarrow x = -1, 1, 2$$

Thus, the given curve cuts the X-axis at points $(-1, 0)$, $(1, 0)$ and $(2, 0)$. **(1)**

On differentiating both sides of Eq. (i) w.r.t. x, we get

$$\frac{dy}{dx} = (x^2 - 1) \cdot 1 + (x - 2) \cdot 2x$$

[by using product rule of derivative]

$$\Rightarrow \frac{dy}{dx} = x^2 - 1 + 2x^2 - 4x$$

$$\Rightarrow \frac{dy}{dx} = 3x^2 - 4x - 1 \quad \textbf{(1)}$$

Now, slope of tangent at $(-1, 0)$ is

$$m_1 = \left(\frac{dy}{dx}\right)_{(-1, 0)} = 3(-1)^2 - 4(-1) - 1$$

$$= 3 + 4 - 1 = 6$$

Slope of tangent at $(1, 0)$ is

$$m_2 = \left(\frac{dy}{dx}\right)_{(1, 0)} = 3(1)^2 - 4(1) - 1$$

$$= 3 - 4 - 1 = -2$$

Slope of tangent at $(2, 0)$ is

$$m_3 = \left(\frac{dy}{dx}\right)_{(2, 0)} = 3(2)^2 - 4(2) - 1$$

$$= 12 - 8 - 1 = 3 \quad \textbf{(1)}$$

We know that, equation of tangent at the point (x_1, y_1) is given by $y - y_1 = m(x - x_1)$.

Here, we get three equations of tangents.

Equation of tangent at point $(-1, 0)$ having slope $(m_1) = 6$, is

$$y - 0 = 6(x + 1)$$

$$\Rightarrow \quad y = 6x + 6 \Rightarrow 6x - y = -6$$

Equation of tangent at point $(1, 0)$ having slope $(m_2) = -2$, is

$$y - 0 = -2(x - 1)$$

$$\Rightarrow \quad y = -2x + 2$$

$$\Rightarrow \quad 2x + y = 2$$

and equation of tangent at point $(2, 0)$ having slope $(m_3) = 3$, is

$$y - 0 = 3(x - 2)$$

$$\Rightarrow \quad y = 3x - 6$$

$$\therefore \quad 3x - y = 6 \quad \textbf{(1)}$$

16. Given equation of curve is

$$4x^2 + 9y^2 = 36$$

On differentiating both sides w.r.t. x, we get

$$8x + 18y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad 18y\frac{dy}{dx} = -8x$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{8x}{18y}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-4x}{9y} \quad \text{...(i)} \ \textbf{(1)}$$

But given that, tangent passes through the point $(3\cos\theta, 2\sin\theta)$.

$\therefore$ Slope of the tangent, m

$$\left(\frac{dy}{dx}\right)_{(3\cos\theta,\, 2\sin\theta)} = \frac{-12\cos\theta}{18\sin\theta}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2\cos\theta}{3\sin\theta} \quad \textbf{(1)}$$

Now, equation of tangent at the point $(3\cos\theta, 2\sin\theta)$ having slope, $m = -\dfrac{2\cos\theta}{3\sin\theta}$ is

$$\Rightarrow \quad y - 2\sin\theta = \frac{-2\cos\theta}{3\sin\theta}(x - 3\cos\theta)$$

$[\because y - y_1 = m(x - x_1)]$ (1)

$\Rightarrow 3y\sin\theta - 6\sin^2\theta = -2x\cos\theta + 6\cos^2\theta$

$\Rightarrow 2x\cos\theta + 3y\sin\theta - 6(\sin^2\theta + \cos^2\theta) = 0$

$\therefore 2x\cos\theta + 3y\sin\theta - 6 = 0$ $[\because \sin^2\theta + \cos^2\theta = 1]$

which is the required equation of tangent. (1)

17. Given equation of curve is

$$y = x^4 - 6x^3 + 13x^2 - 10x + 5$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10 \quad (1)$$

Slope of a tangent at point $(1,0)$ is

$$m = \left[\frac{dy}{dx}\right]_{x=1} = 4 - 18 + 26 - 10 = 2 \quad (1)$$

$\therefore$ Equation of tangent at point $(1,0)$ having slope 2 is (1)

$$y - 0 = 2(x - 1)$$

$$\Rightarrow \quad y = 2x - 2$$

Hence, required equation of tangent is $2x - y = 2$ (1)

18. We have to find the points on the given curve where the tangent is parallel to X-axis. We know that, when a tangent is parallel to X-axis, then

$$\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{d}{dx}(x^2 - 2x)^2 = 0$$

$$\Rightarrow \quad 2(x^2 - 2x)(2x - 2) = 0$$

$$\Rightarrow \quad x = 0, 1, 2$$

When $x = 0$, then $y = [0(-2)]^2 = 0$

When $x = 1$, then $y = [1 - 2(1)]^2 = 1$

When $x = 2$, then $y = [2^2 - 2 \times 2]^2 = 0$

Hence, the tangent is parallel to X-axis at the points $(0, 0)$, $(1, 1)$ and $(2, 0)$.

19. Given equation of curve is

$$y = \frac{x - 7}{x^2 - 5x + 6} \quad \text{...(i)}$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(x^2 - 5x + 6)\cdot 1 - (x - 7)(2x - 5)}{(x^2 - 5x + 6)^2}$$

$$\left[\because \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\left[(x^2 - 5x + 6) - y(x^2 - 5x + 6)(2x - 5)\right]}{(x^2 - 5x + 6)^2}$$

$$\left[\begin{array}{l} \because \quad y = \dfrac{x - 7}{x^2 - 5x + 6} \\ \therefore (x - 7) = y(x^2 - 5x + 6) \end{array}\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1 - (2x - 5)y}{x^2 - 5x + 6} \quad \text{...(ii) (1)}$$

[dividing numerator and denominator by $x^2 - 5x + 6$]

Also, given that curve cuts X-axis, so its y-coordinate is zero.

Put $y = 0$ in Eq. (i), we get

$$\frac{x - 7}{x^2 - 5x + 6} = 0$$

$$\Rightarrow \quad x = 7 \quad (1)$$

So, curve passes through the point $(7, 0)$.

Now, slope of tangent at $(7,0)$ is

$$m = \left(\frac{dy}{dx}\right)_{(7,0)} = \frac{1 - 0}{49 - 35 + 6} = \frac{1}{20} \quad (1)$$

Hence, the required equation of tangent passing through the point $(7, 0)$ having slope $1/20$ is

$$y - 0 = \frac{1}{20}(x - 7)$$

$$\Rightarrow \quad 20y = x - 7$$

$$\therefore \quad x - 20y = 7 \quad (1)$$

20. First, find the slope of normal to curve, i.e. $\dfrac{-1}{dy/dx}$ and put them equal to the slope of line, simplify them and find the values of x and y. Further, use the equation $y - y_1 =$ Slope of normal $(x - x_1)$.

Given equation of curve is

$$y = x^3 + 2x + 6 \quad \text{...(i)}$$

and the given equation of line is

$$x + 14y + 4 = 0$$

On differentiating both sides of Eq. (i) w.r.t. x, we get

$$\frac{dy}{dx} = 3x^2 + 2$$

$\therefore$ Slope of normal $= \frac{-1}{\left(\frac{dy}{dx}\right)} = \frac{-1}{3x^2+2}$

Also, slope of the line $x + 14y + 4 = 0$ is $-\frac{1}{14}$. **(1)**

$\left[\because \text{slope of the line } Ax + By + C = 0 \text{ is } -\frac{A}{B}\right]$

We know that, if two lines are parallel, then their slopes are equal.

$$\therefore \quad -\frac{1}{3x^2+2} = -\frac{1}{14}$$

$$\Rightarrow \quad 3x^2 + 2 = 14$$

$$\Rightarrow \quad 3x^2 = 12 \Rightarrow x^2 = 4$$

$$\Rightarrow \quad x = \pm 2$$ **(1)**

When $x = 2$, then from Eq. (i),

$$y = (2)^3 + 2(2) + 6 = 8 + 4 + 6 = 18$$

and when $x = -2$, then from Eq. (i),

$$y = (-2)^3 + 2(-2) + 6 = -8 - 4 + 6 = -6$$

$\therefore$ Normal passes through $(2, 18)$ and $(-2, -6)$.

Also, slope of normal $= \frac{-1}{14}$.

Hence, equation of normal at point $(2, 18)$ is

$$y - 18 = \frac{-1}{14}(x - 2)$$

$$\Rightarrow \quad 14y - 252 = -x + 2$$

$$\Rightarrow \quad x + 14y = 254$$ **(1/2)**

and equation of normal at point $(-2, -6)$ is

$$y + 6 = -\frac{1}{14}(x + 2)$$

$$\Rightarrow \quad 14y + 84 = -x - 2$$

$$\Rightarrow \quad x + 14y = -86$$ **(1/2)**

Hence, the two equations of normal are $x + 14y = 254$ and $x + 14y = -86$. **(1)**

21. Given equations of curves are

$$y^2 = 4ax \quad \ldots\text{(i)}$$

and

$$x^2 = 4by \quad \ldots\text{(ii)}$$

Clearly, the angle of intersection of curves (i) and (ii) is the angle between the tangents to the curves at the point of intersection. **(1/2)**

So, let us first find the intersection point of given curves.

On substituting the value of y from Eq. (ii) in Eq. (i), we get

$$\left(\frac{x^2}{4b}\right)^2 = 4ax$$

$$\Rightarrow \quad \frac{x^4}{16b^2} = 4ax \Rightarrow x^4 = 64ab^2x$$

$$\Rightarrow \quad x^4 - 64ab^2x = 0$$

$$\Rightarrow \quad x(x^3 - 64ab^2) = 0$$

$$\Rightarrow \quad x = 0 \text{ or } x = 4a^{1/3}b^{2/3}$$

Clearly, when $x = 0$, then from Eq. (i), $y = 0$ and when $x = 4a^{1/3}b^{2/3}$, then from Eq. (i),

$$y^2 = 16a^{4/3}b^{2/3} \Rightarrow y = 4a^{2/3}b^{1/3}$$

Thus, the points of intersection are $(0, 0)$ and $(4a^{1/3}b^{2/3}, 4a^{2/3}b^{1/3})$. **(1)**

Now, let us find the angle of intersection at $(0, 0)$ and $(4a^{1/3}b^{2/3}, 4a^{2/3}b^{1/3})$.

Let m_1 be the slope of tangent to the curve (i) and m_2 be the slope of tangent to the curve (ii). **(1/2)**

Angle of intersection at (0, 0)

Now, $m_1 = \left(\frac{dy}{dx}\right)_{at\,(0,0)} = \left(\frac{2a}{y}\right)_{at\,(0,0)} = \infty$

and $m_2 = \left(\frac{dy}{dx}\right)_{at\,(0,0)} = \left(\frac{x}{2b}\right)_{at\,(0,0)} = 0$ **(1)**

$\Rightarrow$ Tangent to the curve (i) is parallel to Y-axis and tangent to the curve (ii) is parallel to X-axis.

$\therefore$ Angle between these curves two is $\frac{\pi}{2}$.

$\Rightarrow$ The angle of intersection of the curves is $\frac{\pi}{2}$. **(1)**

Angle of intersection at $(4a^{1/3}b^{2/3}, 4a^{2/3}b^{1/3})$

Here, $m_1 = \frac{2a}{4a^{2/3}b^{1/3}} = \frac{1}{2}\cdot\frac{a^{1/3}}{b^{1/3}} = \frac{1}{2}\left(\frac{a}{b}\right)^{1/3}$

and $m_2 = \frac{4a^{1/3}b^{2/3}}{2b} = 2\left(\frac{a}{b}\right)^{1/3}$ **(1/2)**

Let θ be the angle between the tangents. Then,

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$$ **(1/2)**

$$= \left|\frac{2\left(\frac{a}{b}\right)^{1/3} - \frac{1}{2}\left(\frac{a}{b}\right)^{1/3}}{1 + 2\left(\frac{a}{b}\right)^{1/3}\frac{1}{2}\left(\frac{a}{b}\right)^{1/3}}\right|$$

$$= \left|\frac{\frac{3}{2}\left(\frac{a}{b}\right)^{1/3}}{1+\left(\frac{a}{b}\right)^{2/3}}\right| = \left|\frac{\frac{3}{2}\left(\frac{a}{b}\right)^{1/3} \cdot b^{2/3}}{b^{2/3}+a^{2/3}}\right| = \left|\frac{3\,(ab)^{1/3}}{2\,(a^{2/3}+b^{2/3})}\right|$$

$$\Rightarrow \theta = \tan^{-1}\left\{\frac{3\,(ab)^{1/3}}{2\,(a^{2/3}+b^{2/3})}\right\}$$

Hence, the angles of intersection of the curves are $\frac{\pi}{2}$ and $\tan^{-1}\left\{\frac{3\,(ab)^{1/3}}{2\,(a^{2/3}+b^{2/3})}\right\}$. **(1)**

22. Given equation of curve is

$y = \cos(x+y), -2\pi \le x \le 2\pi$ ...(i)

and equation of line is $x + 2y = 0$ ...(ii)

$$\Rightarrow \quad y = -\frac{1}{2}x$$

Clearly, slope of tangents to the curve is $\frac{-1}{2}$. **(1/2)**

Let (x_1, y_1) be the point of contact, then we have

$y_1 = \cos(x_1 + y_1)$ [from Eq. (i)] ...(iii) **(1/2)**

and slope of tangent $= \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$

$$= \frac{-\sin(x_1+y_1)}{1+\sin(x_1+y_1)} \quad \textbf{(1)}$$

$$\left[\begin{aligned} &\because \frac{dy}{dx} = -\sin(x+y)\left(1+\frac{dy}{dx}\right) \\ &\Rightarrow \frac{dy}{dx} + \sin(x+y)\frac{dy}{dx} = -\sin(x+y) \\ &\Rightarrow \frac{dy}{dx} = \frac{-\sin(x+y)}{1+\sin(x+y)} \end{aligned}\right]$$

$$\Rightarrow \quad \frac{-1}{2} = \frac{-\sin(x_1+y_1)}{1+\sin(x_1+y_1)} \quad \textbf{(1/2)}$$

$\Rightarrow 1 + \sin(x_1+y_1) = 2\sin(x_1+y_1)$

$\Rightarrow \sin(x_1+y_1) = 1$ ...(iv) **(1/2)**

On squaring and adding Eqs. (iii) and (iv), we get

$\cos^2(x_1+y_1) + \sin^2(x_1+y_1) = 1 + y_1^2$

$\Rightarrow 1 + y_1^2 = 1 \Rightarrow y_1^2 = 0 \Rightarrow y_1 = 0$ **(1)**

On putting $y_1 = 0$ in Eq. (iii), we get $\cos x_1 = 0$

$\Rightarrow x_1 = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{-\pi}{2}, \frac{-3\pi}{2}$ $[\because -2\pi \le x \le 2\pi]$

But only $x_1 = \frac{\pi}{2}$ and $\frac{-3\pi}{2}$ satisfy Eq. (iv). **(1)**

Hence, the points of contact are $\left(\frac{\pi}{2}, 0\right)$ and $\left(\frac{-3\pi}{2}, 0\right)$

$\therefore$ Equations of tangents are

$$y - 0 = \frac{-1}{2}\left(x - \frac{\pi}{2}\right)$$

and $$y - 0 = -\frac{1}{2}\left(x + \frac{3\pi}{2}\right)$$

$\therefore 2x + 4y = \pi$ and $2x + 4y = -3\pi$ **(1)**

23. Given equations of curves are

$$x^2 = 9p\,(9-y) \quad ...(i)$$

and $$x^2 = p\,(y+1) \quad ...(ii)$$

As, these curves cut each other at right angle, therefore their tangent at point of intersection are perpendicular to each other.

So, let us first find the point of intersection and slope of tangents to the curves. **(1)**

From Eqs. (i) and (ii), we get

$$9p\,(9-y) = p\,(y+1)$$

$\therefore$ $$9\,(9-y) = y+1$$

[$\because p \ne 0$, as if $p = 0$, then curves becomes straight, which will be parallel]

$\Rightarrow 81 - 9y = y + 1 \Rightarrow 80 = 10y \Rightarrow y = 8$ **(1)**

On substituting the value of y in Eq. (i), we get

$$x^2 = 9p \quad \Rightarrow \quad x = \pm 3\sqrt{p}$$

Thus, the point of intersection are $(3\sqrt{p}, 8)$ and $(-3\sqrt{p}, 8)$.

Now, consider Eq.(i), we get

$$\frac{x^2}{9p} = 9 - y \Rightarrow y = 9 - \frac{x^2}{9p} \quad \textbf{(1)}$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{-2x}{9p} \quad ...(iii)$$

From Eq. (ii), we get $\frac{x^2}{p} = y + 1$

$$\Rightarrow \quad y = \frac{x^2}{p} - 1$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{2x}{p} \quad ...(iv) \ \textbf{(1)}$$

Now, for intersection point $(3\sqrt{p}, 8)$, we have slope of tangent to the first curve

$$= \frac{-2(3\sqrt{p})}{9p} = \frac{-6\sqrt{p}}{9p} \quad \text{[using Eq. (iii)]}$$

and slope of tangent to the second curve

$$= \frac{2(3\sqrt{p})}{p} = \frac{6\sqrt{p}}{p} \quad \text{[using Eq. (iv)]}$$

∵ Tangents are perpendicular to each other. Then,

Slope of first curve × Slope of second curve = − 1

$\therefore \quad \frac{-6\sqrt{p}}{9p} \times \frac{6\sqrt{p}}{p} = -1 \Rightarrow \frac{4}{p} = 1 \Rightarrow p = 4$ **(1)**

And for intersection point $(-3\sqrt{p}, 8)$, we have slope of tangent to the first curve

$$= \frac{-2(-3\sqrt{p})}{9p} = \frac{6\sqrt{p}}{9p} \quad \text{[using Eq. (iii)]}$$

and slope of tangent to the second curve

$$= \frac{2(-3\sqrt{p})}{p} = \frac{-6\sqrt{p}}{p} \quad \text{[using Eq. (iv)]}$$

∵ Tangents are perpendicular to each other. Then,

$$\frac{6\sqrt{p}}{9p} \times \frac{-6\sqrt{p}}{p} = -1 \quad [\because m_1 m_2 = -1]$$

$$\Rightarrow \quad \frac{4}{p} = 1 \Rightarrow p = 4$$

Hence, the value of p is 4. **(1)**

24. Given equation of curve is

$$y = x^2 - 2x + 7 \quad \text{...(i)}$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 2x - 2$$ **(1)**

(i) Given, equation of the line is $2x - y + 9 = 0$

$$\Rightarrow \quad y = 2x + 9$$

which is of the form $y = mx + c$.

∴ Slope of the line is $m = 2$

If a tangent is parallel to the line, then slope of tangent is equal to the slope of the line.

Therefore, $\frac{dy}{dx} = m \Rightarrow 2x - 2 = 2 \Rightarrow x = 2$

When $x = 2$, then from Eq. (i), we get

$$y = 2^2 - 2 \times 2 + 7 \Rightarrow y = 7$$ **(1)**

The point on the given curve at which tangent is parallel to given line is (2, 7) and the equation of the tangent is

$$y - 7 = 2(x - 2) \quad [\because y - y_1 = m(x - x_1)]$$

$\Rightarrow 2x - y + 3 = 0$

Hence, the equation of the tangent line to the given curve which is parallel to line $2x - y + 9 = 0$ is $y - 2x - 3 = 0$. **(1)**

(ii) The equation of the given line is

$5y - 15x = 13$

$$\Rightarrow \quad y = \frac{15x + 13}{5} = 3x + \frac{13}{5}$$

which is of the form $y = mx + c$.

∴ Slope of the given line is 3. **(1/2)**

If a tangent is perpendicular to the line $5y - 15x = 13$.

Then, the slope of the tangent $= -\frac{1}{3}$

$$\therefore \quad 2x - 2 = \frac{-1}{3} \Rightarrow 2x = \frac{5}{3} \Rightarrow x = \frac{5}{6}$$

When $x = \frac{5}{6}$, then from Eq. (i), we get

$$y = \left(\frac{5}{6}\right)^2 - 2\left(\frac{5}{6}\right) + 7$$

$$= \frac{25}{36} - \frac{10}{6} + 7 = \frac{25 - 60 + 252}{36}$$

$$\Rightarrow \quad y = \frac{217}{36}$$ **(1)**

∴ The point on the given curve at which tangent is perpendicular to given line is $\left(\frac{5}{6}, \frac{217}{36}\right)$ and the equation of the tangent is

$$y - \frac{217}{36} = \frac{-1}{3}\left(x - \frac{5}{6}\right)$$

$$[\because y - y_1 = m(x - x_1)]$$

$$\Rightarrow \quad \frac{36y - 217}{36} = \frac{-x}{3} + \frac{5}{18}$$

$$\Rightarrow \quad \frac{36y - 217}{36} = \frac{-12x + 10}{36}$$

$\therefore \quad 12x + 36y - 227 = 0$ **(1)**

Hence, the equation of the tangent line to the given curve which is perpendicular to the line $5y - 15x = 13$ is $36y + 12x - 227 = 0$. **(1/2)**

25. Given curve is $x^2 = 4y$.

On differentiating both sides w.r.t. x, we get

$$2x = 4\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{x}{2}$$ **(1)**

Let (h, k) be the coordinates of the point of contact of the normal to the curve $x^2 = 4y$. Then, slope of the tangent at (h, k) is given by

$$\left(\frac{dy}{dx}\right)_{(h,k)} = \frac{h}{2}$$

and slope of the normal at $(h, k) = \frac{-1}{dy/dx} = \frac{-2}{h}$.

Therefore, the equation of normal at (h, k) is

$$y - k = \frac{-2}{h}(x - h) \quad \text{...(i)}$$ **(1)**

$$\left[\because \text{equation of normal in slope form is } y - y_1 = -\frac{1}{m}(x - x_1)\right]$$

Since, it passes through the point (1, 2), so on putting $x = 1$ and $y = 2$, we get

$$2 - k = \frac{-2}{h}(1 - h)$$

$$\Rightarrow \quad k = 2 + \frac{2}{h}(1 - h) \quad \text{...(ii)} \quad \text{(1)}$$

Since, (h, k) also lies on the curve $x^2 = 4y$,

therefore $h^2 = 4k$ ...(iii) **(1)**

On solving Eqs.(ii) and (iii), we get

$$h = 2 \text{ and } k = 1$$

Substituting the values of h and k in Eq.(i), the required equation of normal is

$$y - 1 = \frac{-2}{2}(x - 2) \Rightarrow x + y = 3 \quad \text{(1)}$$

Now, equation of tangent at (h, k) is

$$y - k = \frac{h}{2}(x - h)$$

On putting $h = 2$ and $k = 1$, we get

$$y - 1 = \frac{2}{2}(x - 2) \Rightarrow y - 1 = x - 2$$

$$\therefore \quad y = x - 1 \quad \text{(1)}$$

26. First, differentiate the given curve with respect to x and determine $\frac{dy}{dx}$. Then, find the equation of tangent at (x_1, y_1). Now, as this equation is passes through given point (x_0, y_0), so this point will satisfy the tangent and curve also. Further, simplify it and get the required equations of tangent.

Given equation of curve is

$$3x^2 - y^2 = 8 \quad \text{...(i)}$$

On differentiating both sides w.r.t. x, we get

$$6x - 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{3x}{y} \quad \text{(1)}$$

Equation of tangent at point (h, k) is

$$y - k = \left(\frac{dy}{dx}\right)_{(h,k)}(x - h)$$

$$\Rightarrow \quad y - k = \frac{3h}{k}(x - h) \quad \text{...(ii)} \quad \text{(1)}$$

Since, it is passes through the point $\left(\frac{4}{3}, 0\right)$

$$\therefore \quad 0 - k = \frac{3h}{k}\left(\frac{4}{3} - h\right) \Rightarrow -k^2 = 3h\frac{(4 - 3h)}{3}$$

$$\Rightarrow \quad 3h^2 - k^2 - 4h = 0 \quad \text{...(iii)} \quad \text{(1)}$$

Also, the point (h, k) satisfy the Eq. (i), so we get

$$3h^2 - k^2 = 8 \quad \text{...(iv)}$$

Now, on solving Eqs. (iii) and (iv), we get

$$4h = 8$$

$$\Rightarrow \quad h = 2$$

On putting $h = 2$ in Eq. (iv), we get

$$3(2)^2 - k^2 = 8$$

$$\Rightarrow \quad k^2 = 4$$

$$\Rightarrow \quad k = \pm 2 \quad \text{(1)}$$

Now, putting the values of h and k in Eq. (ii), we get

$$y - (\pm 2) = \frac{3(2)}{\pm 2}(x - 2)$$

$$\Rightarrow \quad y \mp 2 = \pm 3(x - 2)$$

$$\Rightarrow \quad y = \pm 3(x - 2) \pm 2$$

$$\Rightarrow \quad y = \pm\{3(x - 2) + 2\} \quad \text{(1)}$$

$$\Rightarrow \quad y = \pm(3x - 6 + 2)$$

$$\Rightarrow \quad y = \pm(3x - 4)$$

Hence, $y = -3x + 4$ and $y = 3x - 4$ are two required equations of tangent. **(1)**

27. Given curve is $y = 4x^3 - 2x^5$. ...(i)

Let any point on the curve be (x_1, y_1). So, it satisfies Eq. (i).

$$\therefore \quad y_1 = 4x_1^3 - 2x_1^5 \quad \text{...(ii)}$$

On differentiating both sides of Eq. (i), we get

$$\frac{dy}{dx} = 12x^2 - 10x^4 \quad \text{(1/2)}$$

Equation of tangent at point (x_1, y_1) is

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}(x - x_1)$$

$$\Rightarrow \quad y - y_1 = [12(x_1)^2 - 10(x_1)^4](x - x_1) \quad \text{(1)}$$

Since, it passes through the origin.

$$\therefore \quad 0 - y_1 = (12x_1^2 - 10x_1^4)(0 - x_1)$$

$$\Rightarrow \quad y_1 = (12x_1^2 - 10x_1^4)x_1 \quad \text{...(iii)} \quad \text{(1/2)}$$

From Eqs. (ii) and (iii), we get

$$(12x_1^2 - 10x_1^4)x_1 = 4x_1^3 - 2x_1^5$$

$$\Rightarrow \quad 2x_1^3(6 - 5x_1^2) = 2x_1^3(2 - x_1^2)$$

$$\Rightarrow \quad 2x_1^3(4 - 4x_1^2) = 0$$

$$\Rightarrow \quad x_1 = 0 \text{ or } 4 - 4x_1^2 = 0$$

$$\therefore \quad x_1 = 0 \text{ or } x_1 = \pm 1 \quad \text{(1)}$$

On putting the values of $x_1 = 0, 1$ and -1 respectively in Eq. (ii), we get

At $x_1 = 0$, $y_1 = 0$

At $x_1 = 1$, $y_1 = 4(1)^3 - 2(1)^5 = 4 - 2 = 2$ (1)

and at $x_1 = (-1)$,

$$y_1 = 4(-1)^3 - 2(-1)^5$$

$$= 4(-1) - 2(-1) = -4 + 2 = -2$$ (1)

Hence, all points on the curve at which the tangent passes through origin, are (0, 0), (1, 2) and (−1, −2). (1)

28. Given equations of curves are

$$x = y^2 \quad \ldots(i)$$

and $$xy = k \quad \ldots(ii)$$

Let the curves (i) and (ii) cut at right angle. (1/2)

Let us first find the point of intersection of given curves.

On substituting the value of x from Eq. (i) in Eq. (ii), we get

$$y^3 = k \Rightarrow y = k^{1/3}$$

On substituting $y = k^{1/3}$ in Eq. (ii), we get

$$x = k^{2/3}$$

Thus, the point of intersection is $(k^{1/3}, k^{2/3})$. (1)

Now, let m_1 be the slope of tangent to the curve (i) and m_2 be the slope of tangent to the curvet (ii).

Then, $m_1 = \left(\frac{dy}{dx}\right)_{(k^{2/3}, k^{1/3})} = \left(\frac{1}{2y}\right)_{(k^{2/3}, k^{1/3})} = \frac{1}{2k^{1/3}}$

$\left[\text{from Eq. (i), we have } 1 = 2y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2y}\right]$ (1)

and $m_2 = \left(\frac{dy}{dx}\right)_{(k^{2/3}, k^{2/3})} = \left(-\frac{y}{x}\right)_{(k^{2/3}, k^{1/3})}$

$$= \frac{-k^{1/3}}{k^{2/3}} = \frac{-1}{k^{1/3}}$$

$\left[\text{from Eq. (ii), we have } x\frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}\right]$ (1)

Since, the curves (i) and (ii) cut at right angle.

∴ Angle between the tangents to the curves at the point of intersection is $\frac{\pi}{2}$, i.e. tangents to the curves are perpendicular to each other.

$\Rightarrow \quad m_1 \cdot m_2 = -1$ (1)

$\Rightarrow \quad \left(\frac{1}{2k^{1/3}}\right)\left(\frac{-1}{k^{1/3}}\right) = -1$

$\Rightarrow \quad \frac{1}{2k^{2/3}} = 1 \Rightarrow 2k^{2/3} = 1$ (1)

On cubing both sides, we get

$8k^2 = 1$ **Hence proved.** (1/2)

29. Given equations of curves are

$$x = a\cos t + at\sin t \text{ and } y = a\sin t - at\cos t$$

On differentiating x and y separately w.r.t. t, we get

$$\frac{dx}{dt} = -a\sin t + a(t\cos t + \sin t)$$

$$= -a\sin t + at\cos t + a\sin t = at\cos t$$

and $\frac{dy}{dt} = a\cos t - a[t(-\sin t) + \cos t)]$

$$= a\cos t + at\sin t - a\cos t = at\sin t$$ (1)

Now, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{at\sin t}{at\cos t} = \tan t$ (1/2)

$\Rightarrow$ Slope of normal at any point $t = \frac{-1}{\tan t} = -\cot t$ (1)

Now, equation of normal at any point t is given by

$y - (a\sin t - at\cos t) = -\cot t\ [x - (a\cos t + at\sin t)]$ (1)

$\Rightarrow y - a\sin t + at\cos t = \frac{-\cos t}{\sin t}(x - a\cos t - at\sin t)$

$\Rightarrow y\sin t - a\sin^2 t + at\cos t\sin t = -x\cos t + a\cos^2 t + at\sin t\cos t$

$\Rightarrow \quad x\cos t + y\sin t = a(\cos^2 t + \sin^2 t)$ (1)

$\Rightarrow \quad x\cos t + y\sin t = a \quad [\because \sin^2\theta + \cos^2\theta = 1]$

$\therefore \quad x\cos t + y\sin t - a = 0$ (1/2)

Now, the distance of normal from the origin is given by

$$\frac{|0\cdot\cos t + 0\cdot\sin t - a|}{\sqrt{\cos^2 t + \sin^2 t}} = \frac{|-a|}{\sqrt{1}} = a$$ (1)

Hence proved.

30.

> First, differentiate the given curve with respect to θ and then determine $\frac{dy}{dx}$ at $\theta = \frac{\pi}{4}$. Further, use the formula, equation of tangent at (x_1, y_1) is $y - y_1 = m(x - x_1)$
>
> and equation of normal at (x_1, y_1) is
>
> $$y - y_1 = -\frac{1}{m}(x - x_1).$$

Given curves are $x = 1 - \cos\theta$ and $y = \theta - \sin\theta$.

On differentiating x and y separately w.r.t. θ, we get

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(1 - \cos\theta) = \sin\theta$$

and $$\frac{dy}{d\theta} = \frac{d}{d\theta}(\theta - \sin\theta) = 1 - \cos\theta$$

Now, $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{1-\cos\theta}{\sin\theta}$

At $\theta = \frac{\pi}{4}$, $\left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{4}} = \frac{1-\cos\frac{\pi}{4}}{\sin\frac{\pi}{4}} = \frac{1-\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}$

$= \sqrt{2}-1$ **(1)**

Also, at $\theta=\frac{\pi}{4}$,

$$x_1 = 1-\cos\frac{\pi}{4} = 1-\frac{1}{\sqrt{2}} = \frac{\sqrt{2}-1}{\sqrt{2}}$$

and $y_1 = \frac{\pi}{4} - \sin\frac{\pi}{4} = \frac{\pi}{4} - \frac{1}{\sqrt{2}}$ **(1)**

We know that, equation of tangent at (x_1, y_1) having slope m, is given by

$$y - y_1 = m(x - x_1)$$

$\therefore$ $y-\left(\frac{\pi}{4}-\frac{1}{\sqrt{2}}\right) = (\sqrt{2}-1)\left[x-\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)\right]$

$\Rightarrow$ $y-\frac{\pi}{4}+\frac{1}{\sqrt{2}} = x(\sqrt{2}-1)-\frac{(\sqrt{2}-1)^2}{\sqrt{2}}$ **(1)**

$\Rightarrow$ $y-\frac{\pi}{4}+\frac{1}{\sqrt{2}} = x(\sqrt{2}-1)-\frac{(2+1-2\sqrt{2})}{\sqrt{2}}$

$\Rightarrow$ $\left(y-\frac{\pi}{4}+\frac{1}{\sqrt{2}}\right) = x(\sqrt{2}-1)-\frac{(3-2\sqrt{2})}{\sqrt{2}}$

$\Rightarrow$ $x(\sqrt{2}-1)-y = \frac{3-2\sqrt{2}}{\sqrt{2}}-\frac{\pi}{4}+\frac{1}{\sqrt{2}}$

Hence, the equation of tangent is

$$x(\sqrt{2}-1)-y = \frac{12-8\sqrt{2}-\sqrt{2}\pi+4}{4\sqrt{2}}$$

$\Rightarrow x(8-4\sqrt{2})-4\sqrt{2}y = (16-\sqrt{2}\pi-8\sqrt{2})$ **(1)**

Also, the equation of normal at (x_1, y_1) having slope $-\frac{1}{m}$ is given by

$$y - y_1 = -\frac{1}{m}(x-x_1)$$

$\Rightarrow y-\left(\frac{\pi}{4}-\frac{1}{\sqrt{2}}\right) = \frac{-1}{\sqrt{2}-1}\left(x-\frac{\sqrt{2}-1}{\sqrt{2}}\right)$

$\Rightarrow$ $y(\sqrt{2}-1)-\left(\frac{\sqrt{2}\pi-4}{4\sqrt{2}}\right)(\sqrt{2}-1)$

$= -x+\frac{\sqrt{2}-1}{\sqrt{2}}$ **(1)**

$\Rightarrow$ $y(\sqrt{2}-1)-\left(\frac{2\pi-\sqrt{2}\pi-4\sqrt{2}+4}{4\sqrt{2}}\right)$

$= \frac{-\sqrt{2}x+\sqrt{2}-1}{\sqrt{2}}$

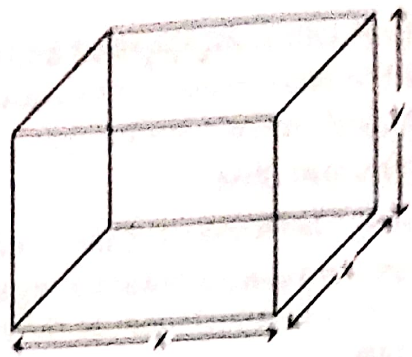
$\Rightarrow$ $4\sqrt{2}y(\sqrt{2}-1)-2\pi+\sqrt{2}\pi+4\sqrt{2}-4$

$= -4\sqrt{2}x+4\sqrt{2}-4$

$\Rightarrow$ $4\sqrt{2}x+4\sqrt{2}y(\sqrt{2}-1) = 2\pi-\sqrt{2}\pi$

$\Rightarrow$ $4\sqrt{2}x+y(8-4\sqrt{2}) = 2\pi-\sqrt{2}\pi$

$\therefore$ $4\sqrt{2}x+(8-4\sqrt{2})y = \pi(2-\sqrt{2})$ **(1)**



Then, $$V = x^2 y \quad \text{...(i)}$$

[∵ volume of cuboid $= L \times B \times H$]

Clearly, the surface area, $S = 4xy + x^2$

$$= 4x \cdot \frac{V}{x^2} + x^2 \quad \text{[using Eq. (i)]}$$

$$\Rightarrow \quad S(x) = \frac{4V}{x} + x^2 \quad (1)$$

Now, on differentiating both sides w.r.t. x, we get

$$S'(x) = \frac{-4V}{x^2} + 2x \quad \text{...(ii)} \quad (1/2)$$

On putting $S'(x) = 0$, we get

$$\frac{-4V}{x^2} + 2x = 0$$

$$\Rightarrow \quad \frac{-4V}{x^2} = -2x$$

$$\Rightarrow \quad x^3 = 2V \Rightarrow x = (2V)^{\frac{1}{3}} \quad (1/2)$$

Again, on differentiating both sides of Eq. (ii) w.r.t. x, we get

$$S''(x) = \frac{8V}{x^3} + 2$$

and $S''((2V)^{1/3}) = \frac{8V}{2V} + 2 = 4 + 2 = 6 > 0$

∴ $S(x)$ is minimum when $x = (2V)^{\frac{1}{3}}$.

From Eq. (i), we get

$$y = \frac{V}{x^2} = \frac{\left(\frac{x^3}{2}\right)}{x^2} = \frac{x}{2} \quad [\because x^3 = 2V]$$

Thus, the cost of material will be least when depth of the tank is half of its width. (2)

Solutions

1. Let x be the length of a side of square base and y be the length of vertical side. Also, let V be the given quantity of water.

2. Let r be the radius of circle and x be the side of a square. Then, given that

Perimeter of square + Perimeter of circle $= k$ (constant) i.e. $4x + 2\pi r = k \Rightarrow x = \dfrac{k - 2\pi r}{4}$ ...(i) **(1)**

Let A denotes the sum of their areas.

$\therefore$ Area, A = area of a square + area of circle

$\therefore \quad A = x^2 + \pi r^2$ ...(ii)

On putting the value of x from Eq. (i) in Eq.(ii), we get

$$A = \left(\frac{k - 2\pi r}{4}\right)^2 + \pi r^2$$

On differentiating both sides w.r.t. r, we get

$$\frac{dA}{dr} = 2\left(\frac{k - 2\pi r}{4}\right)\left(-\frac{2\pi}{4}\right) + 2\pi r$$

$$= -\frac{\pi}{4}(k - 2\pi r) + 2\pi r \quad \textbf{(1)}$$

For maxima and minima, put $\dfrac{dA}{dr} = 0$

$\Rightarrow \quad -\dfrac{\pi}{4}(k - 2\pi r) + 2\pi r = 0$

$\Rightarrow \quad -\dfrac{\pi}{4}k + \dfrac{\pi^2 r}{2} + 2\pi r = 0$

$\Rightarrow \quad \dfrac{r\pi}{2}(\pi + 4) = \dfrac{\pi}{4}k$

$\Rightarrow \quad r = \dfrac{k}{2\pi + 8}$ ...(iii)

Now, $\dfrac{d^2A}{dr^2} = \dfrac{d}{dr}\left(\dfrac{dA}{dr}\right) = \dfrac{d}{dr}\left[2\pi r - \dfrac{\pi}{4}(k - 2\pi r)\right]$

$$= 2\pi + \frac{2\pi^2}{4}$$

$$= 2\pi + \frac{\pi^2}{2} > 0$$

$\because \quad \dfrac{d^2A}{dr^2} > 0$, so A is minimum. **(1)**

From Eq. (iii), we get

$$r = \frac{k}{2\pi + 8}$$

$\Rightarrow \quad 2r\pi + 8r = k$

$\Rightarrow \quad 2r\pi + 8r = 4x + 2\pi r \quad [\because k = 4x + 2\pi r]$

$\therefore \quad 8r = 4x$

$\Rightarrow \quad x = 2r$

i.e. Side of square = double the radius of circle

Hence, sum of area of a circle and a square is least, when side of square is equal to diameter of circle or double the radius of circle. **(1)**

Hence proved.

3. Let x m be the length, y m be the breadth and h=2m be the depth of the tank. Let ₹ H be the total cost for building the tank. Now, given that $h = 2$ m and volume of tank = 8 m³.

Clearly, area of the rectangular base of the tank

= length × breadth = xy m² **(1)**

and the area of the four rectangular sides

= 2 (length + breadth) × height

= $2(x + y) \times 2 = 4(x + y)$ m²

$\therefore$ Total cost, $H = 70 \times xy + 45 \times 4(x + y)$

$\Rightarrow \quad H = 70xy + 180(x + y)$ ...(i) **(1)**

Also, volume of tank = 8 m³

$\Rightarrow l \times b \times h = 8 \Rightarrow x \times y \times 2 = 8 \Rightarrow y = \dfrac{4}{x}$ ...(ii) **(1/2)**

On putting the value of y from Eq. (ii) in Eq. (i), we get

$$H = 70x \times \frac{4}{x} + 180\left(x + \frac{4}{x}\right)$$

$\Rightarrow \quad H = 280 + 180\left(x + \dfrac{4}{x}\right)$ ...(iii) **(1)**

On differentiating both sides w.r.t. x, we get

$$\frac{dH}{dx} = 180\left(1 - \frac{4}{x^2}\right)$$

For maxima or minima, put $\dfrac{dH}{dx} = 0$

$\Rightarrow \quad 180\left(1 - \dfrac{4}{x^2}\right) = 0 \Rightarrow 1 - \dfrac{4}{x^2} = 0$

$\Rightarrow \quad \dfrac{4}{x^2} = 1$

$\Rightarrow \quad x^2 = 4 \Rightarrow x = 2 \quad [\because x > 0]$ **(1)**

Also, $\dfrac{d^2H}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dH}{dx}\right) = \dfrac{d}{dx}\left[180\left(1 - \dfrac{4}{x^2}\right)\right]$

$$= \frac{8}{x^3} \times 180$$

At $x = 2$, $\left[\dfrac{d^2H}{dx^2}\right]_{x=2} = \dfrac{8}{2^3} \times 180 = 180 > 0$

$\because \quad \dfrac{d^2H}{dx^2} > 0$

$\Rightarrow$ H is least at $x = 2$. **(1)**

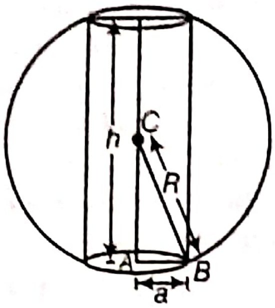
Also, the least cost $= 280 + 180\left(2 + \dfrac{4}{2}\right)$

[put $x = 2$ in Eq. (iii) to get least cost H]

= 280 + 180 × 4 = 280 + 720 = ₹1000

Hence, the cost of least expensive tank is ₹1000. **(1/2)**

4. Let h be the height and a be the radius of base of cylinder inscribed in the given sphere of radius (R).



In ΔABC, $AB^2 + AC^2 = BC^2$

[by Pythagoras theorem]

$$\Rightarrow \quad a^2 + \left(\frac{h}{2}\right)^2 = R^2 \Rightarrow a^2 = R^2 - \frac{h^2}{4} \quad (1)$$

Volume of cylinder, $V = \pi a^2 h$

$$= \pi h\left(R^2 - \frac{h^2}{4}\right) = \frac{\pi}{4}(4R^2h - h^3) \quad (1)$$

On differentiating both sides two times w.r.t. h, we get

$$\frac{dV}{dh} = \frac{\pi}{4}(4R^2 - 3h^2)$$

and $$\frac{d^2V}{dh^2} = \frac{\pi}{4}(-6h) = -\frac{3\pi h}{2} \quad \text{...(i) (1)}$$

For maxima or minima, put $\frac{dV}{dh} = 0$.

$$\Rightarrow \quad \frac{\pi}{4}(4R^2 - 3h^2) = 0 \Rightarrow h^2 = \frac{4}{3}R^2$$

$$\Rightarrow \quad h = \frac{2}{\sqrt{3}}R \quad (1)$$

[$\because$ height is always positive, so we do not take '–' sign]

On substituting the value of h in Eq. (i), we get

$$\frac{d^2V}{dh^2} = \frac{-3\pi}{2} \cdot \frac{2}{\sqrt{3}}R = -\sqrt{3}\pi R < 0$$

$\Rightarrow V$ is maximum.

Hence, the required height of cylinder is $\frac{2R}{\sqrt{3}}$. (1)

Hence proved.

Now, maximum volume of cylinder, V

$$= \pi h\left(R^2 - \frac{h^2}{4}\right) = \pi \frac{2R}{\sqrt{3}}\left(R^2 - \frac{1}{4} \cdot \frac{4}{3}R^2\right)$$

$$\left[\text{put } h = \frac{2}{\sqrt{3}}R\right]$$

$$= \frac{2\pi R}{\sqrt{3}} \frac{(3R^2 - R^2)}{3} = \frac{4\pi R^3}{3\sqrt{3}} \text{ cu units} \quad (1)$$

5. Given, equation of curve is $y^2 = 4x$.

Let $P(x, y)$ be a point on the curve, which is nearest to point $A(2, -8)$. (1)

Now, distance between the points A and P is given by

$$AP = \sqrt{(x-2)^2 + (y+8)^2}$$

$$= \sqrt{\left(\frac{y^2}{4} - 2\right)^2 + (y+8)^2}$$

$$= \sqrt{\frac{y^4}{16} + 4 - y^2 + y^2 + 16y + 64}$$

$$= \sqrt{\frac{y^4}{16} + 16y + 68} \quad (1)$$

Let $z = AP^2 = \frac{y^4}{16} + 16y + 68$

Now, $\frac{dz}{dy} = \frac{1}{16} \times 4y^3 + 16 = \frac{y^3}{4} + 16$ (1)

For maximum or minimum value of z, put

$$\frac{dz}{dy} = 0 \Rightarrow \frac{y^3}{4} + 16 = 0$$

$$\Rightarrow y^3 + 64 = 0 \Rightarrow (y+4)(y^2 - 4y + 16) = 0$$

$$\Rightarrow \quad y = -4 \quad (1)$$

[$\because y^2 - 4y + 16 = 0$ gives imaginary values of y]

Now, $\frac{d^2z}{dy^2} = \frac{1}{4} \times 3y^2 = \frac{3}{4}y^2$

For $y = -4$,

$$\frac{d^2z}{dy^2} = \frac{3}{4}(-4)^2 = 12 > 0 \quad (1)$$

Thus, z is minimum when $y = -4$.

Substituting $y = -4$ in equation of the curve $y^2 = 4x$; we obtain $x = 4$.

Hence, the point $(4, -4)$ on the curve $y^2 = 4x$ is nearest to the point $(2, -8)$. (1)

6. Let R be the radius and h be the height of the cone, which inscribed in a sphere of radius r.

$$\therefore \quad OA = h - r$$

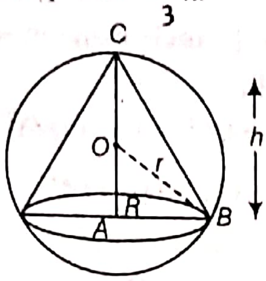
In ΔOAB, by Pythagoras theorem, we have

$$r^2 = R^2 + (h-r)^2$$

$$\Rightarrow \quad r^2 = R^2 + h^2 + r^2 - 2rh$$

$$\Rightarrow \quad R^2 = 2rh - h^2 \quad \text{...(i) (1)}$$

The volume of sphere $= \frac{4}{3}\pi r^3$



(1)

and the volume V of the cone,

$$V = \frac{1}{3}\pi R^2 h$$

$$\Rightarrow \quad V = \frac{1}{3}\pi h\,(2rh - h^2) \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad V = \frac{1}{3}\pi\,(2rh^2 - h^3) \quad \text{...(ii)}$$

On differentiating both sides of Eq. (ii) w.r.t. h, we get

$$\frac{dV}{dh} = \frac{1}{3}\pi\,(4rh - 3h^2) \quad \text{...(iii) (1)}$$

For maxima or minima, put $\frac{dV}{dh} = 0$

$$\Rightarrow \quad \frac{1}{3}\pi(4rh - 3h^2) = 0$$

$$\Rightarrow \quad 4rh = 3h^2$$

$$\Rightarrow \quad 4r = 3h$$

$$\Rightarrow \quad h = \frac{4r}{3} \quad [\because h \neq 0] \quad \text{(1)}$$

Again, on differentiating Eq. (iii) w.r.t. h, we get

$$\frac{d^2V}{dh^2} = \frac{1}{3}\pi\,(4r - 6h)$$

At $h = \frac{4r}{3}$, $\left(\frac{d^2V}{dh^2}\right)_{h=\frac{4r}{3}} = \frac{1}{3}\pi\left(4r - 6 \times \frac{4r}{3}\right)$

$$= \frac{\pi}{3}(4r - 8r) = -\frac{4r\pi}{3} < 0$$

$\Rightarrow V$ is maximum at $h = \frac{4r}{3}$. (1)

Hence proved.

On substituting the value of h in Eq. (ii), we get

$$V = \frac{1}{3}\pi\left[2r\left(\frac{4r}{3}\right)^2 - \left(\frac{4r}{3}\right)^3\right]$$

$$= \frac{\pi}{3}\left[\frac{32}{9}r^3 - \frac{64}{27}r^3\right] = \frac{\pi}{3}r^3\left[\frac{32}{9} - \frac{64}{27}\right]$$

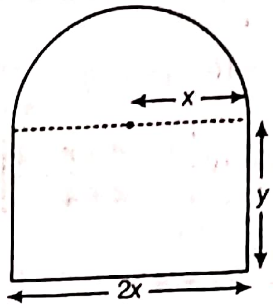
$$= \frac{\pi}{3}r^3\left[\frac{96 - 64}{27}\right] = \frac{\pi}{3}r^3\left(\frac{32}{27}\right)$$

$$= \frac{8}{27} \times \left(\frac{4}{3}\pi r^3\right) = \frac{8}{27} \times \text{(Volume of sphere)}$$

Hence, maximum volume of the cone is $\frac{8}{27}$ of the volume of the sphere. (1)

7. Let $2x$ be the length and y be the width of the window.

Then, radius of semicircular opening $= x$ m



Since, perimeter of the window is 10 m.

$$\therefore \quad 2x + y + y + \frac{2\pi x}{2} = 10$$

$$\Rightarrow \quad 2x + 2y + \pi x = 10$$

$$\Rightarrow \quad x(\pi + 2) + 2y = 10$$

$$\Rightarrow \quad y = \frac{10 - x(\pi + 2)}{2} \quad \text{...(i) (1)}$$

Note that, to admit maximum light, area of window should be maximum.

Here, area of window

A = area of rectangle + area of semicircular region

$$= 2x \times y + \frac{1}{2}\pi x^2 \quad \text{(1)}$$

$$\Rightarrow A = 2x\left(\frac{10 - x(\pi + 2)}{2}\right) + \frac{1}{2}\pi x^2 \quad \text{[from Eq. (i)]}$$

(1/2)

$$\Rightarrow A = 10x - x^2(\pi + 2) + \frac{1}{2}\pi x^2$$

On differentiating both sides w.r.t. x, we get

$$\frac{dA}{dx} = 10 - 2x(\pi + 2) + \pi x$$

$$= 10 - 2x\pi - 4x + \pi x$$

$$= 10 - \pi x - 4x \quad \text{...(ii) (1)}$$

For maximum, Put $\frac{dA}{dx} = 0$

$$\Rightarrow \quad 10 = \pi x + 4x$$

$$\Rightarrow \quad x = \frac{10}{\pi + 4} \quad \text{(1/2)}$$

Again, on differentiating both sides of Eq. (ii), we get

$$\frac{d^2A}{dx^2} = -\pi - 4$$

$$\Rightarrow \quad \left.\frac{d^2A}{dx^2}\right|_{x=\frac{10}{\pi+4}} = -(\pi + 4) < 0$$

Thus, area is maximum when $x = \frac{10}{\pi + 4}$ (1)

Now, on substituting the value of x in Eq. (i), we get

$$2y = 10 - (\pi + 2) \times \frac{10}{\pi + 4}$$

$$\Rightarrow \quad 2y = 10\left[\frac{\pi + 4 - \pi - 2}{\pi + 4}\right] = \frac{20}{\pi + 4}$$

$$\therefore \quad y = \frac{10}{\pi + 4}$$

Hence, length of window $= \frac{20}{\pi + 4}$ m and width of window $= \frac{10}{\pi + 4}$ m, to admit maximum light through the whole opening. (1)

8. Let V be the fixed volume of a closed cuboid with length x, breadth x and height y.

Then, $V = x \times x \times y$

$$\Rightarrow \quad y = \frac{V}{x^2} \quad \text{...(i)} \quad (1)$$

Let S be its surface area.

Then, $S = 2(x^2 + xy + xy)$ (1)

$$\Rightarrow \quad S = 2(x^2 + 2xy) = 2\left(x^2 + \frac{2V}{x}\right) \text{[using Eq. (i)]}$$

$$\Rightarrow \quad S = 2\left(x^2 + \frac{2V}{x}\right)$$

$$\Rightarrow \quad \frac{dS}{dx} = 2\left(2x - \frac{2V}{x^2}\right)$$

and $\frac{d^2S}{dx^2} = \left(4 + \frac{8V}{x^3}\right)$ (1)

Now, $\frac{dS}{dx} = 0$

$$\Rightarrow \quad 2\left(2x - \frac{2V}{x^2}\right) = 0$$

$$\Rightarrow \quad 2x = \frac{2V}{x^2}$$

$$\Rightarrow \quad x^3 = V$$

$$\Rightarrow \quad V = x^3 \quad (1)$$

$$\Rightarrow \quad x \times x \times y = x^3 \Rightarrow y = x \quad (1)$$

Also $\left(\frac{d^2S}{dx^2}\right)_{x = y^{1/3}} = 4 + \frac{8V}{V} = 12 > 0$

So, S is minimum when length $= x$, breadth $= x$ and height $= x$, when it is cube. (1)

9. Let $AC = x$, $BC = y$ and r be the radius of circle.

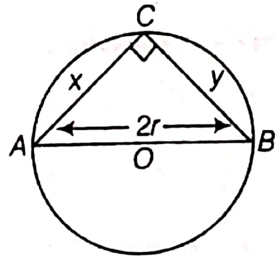
Also, $\angle C = 90°$ [$\because$ angle made in semi-circle is 90°]

In ΔABC, we have

$$(AB)^2 = (AC)^2 + (BC)^2$$

$$\Rightarrow \quad (2r)^2 = (x)^2 + (y)^2$$

$$\Rightarrow \quad 4r^2 = x^2 + y^2 \quad \text{...(i)} \quad (1)$$



We know that,

Area of $\Delta ABC, (A) = \frac{1}{2} x \cdot y$ (1)

On squaring both sides, we get

$$A^2 = \frac{1}{4} x^2 y^2$$

Let $A^2 = S$

Then, $S = \frac{1}{4} x^2 y^2$

$$\Rightarrow \quad S = \frac{1}{4} x^2 (4r^2 - x^2) \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad S = \frac{1}{4}(4x^2 r^2 - x^4) \quad (1)$$

On differentiating both sides w.r.t. x, we get

$$\frac{dS}{dx} = \frac{1}{4}(8r^2 x - 4x^3)$$

For maxima or minima, put $\frac{dS}{dx} = 0$

$$\therefore \quad \frac{1}{4}(8r^2 x - 4x^3) = 0$$

$$\Rightarrow \quad 8r^2 x = 4x^3 \Rightarrow 8r^2 = 4x^2 \Rightarrow x^2 = 2r^2$$

$$\Rightarrow \quad x = \sqrt{2}r \quad (1)$$

From Eq. (i), we get

$$y^2 = 4r^2 - 2r^2 = 2r^2 \Rightarrow y = \sqrt{2}r$$

Here, $x = y$, so triangle is an isosceles. (1)

Also, $\frac{d^2S}{dx^2} = \frac{d}{dx}\left[\frac{1}{4}(8r^2 x - 4x^3)\right] = \frac{1}{4}(8r^2 - 12x^2)$

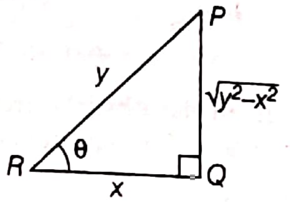
$$= 2r^2 - 3x^2$$

At $x = \sqrt{2}r$, $\frac{d^2S}{dx^2} = 2r^2 - 3(2r^2) < 0 = -4r^2 < 0$

Hence, area is maximum when triangle is an isosceles.

Hence proved. (1)

10. Let us consider a right angled triangle with base = x and hypotenuse = y. Let $x + y = k$, where k is a constant. Let θ be the angle between the base and the hypotenuse. Let A be the area of the triangle, then



$$A = \frac{1}{2} \times QR \times PQ = \frac{1}{2} x \sqrt{y^2 - x^2} \quad \textbf{(1)}$$

$$\Rightarrow \quad A^2 = \frac{x^2 (y^2 - x^2)}{4}$$

$$\Rightarrow \quad A^2 = \frac{x^2[(k-x)^2 - x^2]}{4} \quad [\because y = k - x]$$

$$\Rightarrow \quad A^2 = \frac{k^2x^2 - 2kx^3}{4} \quad \ldots(i) \ \textbf{(1)}$$

$$\Rightarrow \quad 2A \frac{dA}{dx} = \frac{2k^2x - 6kx^2}{4} \quad \ldots(ii)$$

$$\Rightarrow \quad \frac{dA}{dx} = \frac{k^2x - 3kx^2}{4A} \quad \textbf{(1/2)}$$

Now, for maxima or minima

Put $\frac{dA}{dx} = 0 \Rightarrow (k^2x - 3kx^2) = 0 \Rightarrow x = \frac{k}{3}$ **(1/2)**

Again, $2\left(\frac{dA}{dx}\right)^2 + 2A \frac{d^2A}{dx^2} = \frac{2k^2 - 12kx}{4}$ ...(iii)

[using Eq. (ii)]

Put $\frac{dA}{dx} = 0$ and $x = \frac{k}{3}$ in Eq. (iii), we get

$$\frac{d^2A}{dx^2} = \frac{-k^2}{4A} < 0$$

Thus, A is maximum when $x = \left(\frac{k}{3}\right)$. **(1)**

Now, $\quad x = \frac{k}{3}$

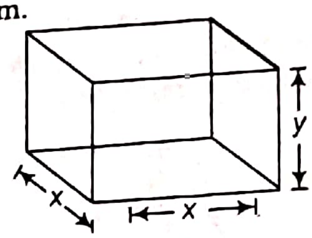
$$\Rightarrow \quad y = \left(k - \frac{k}{3}\right) = \frac{2k}{3}$$

$$\therefore \quad \frac{x}{y} = \cos\theta$$

$$\Rightarrow \quad \cos\theta = \frac{k/3}{2k/3} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

Hence, the area of triangle is maximum, when $\theta = \frac{\pi}{3}$. **Hence proved.** **(1)**

11. Given, volume of the box = 1024 cm^3. Let length of the side of square base be x cm and height of the box be y cm.



Volume of the box, $V = x^2 \cdot y = 1024$ **(1)**

$$\Rightarrow \quad x^2 y = 1024$$

$$\Rightarrow \quad y = \frac{1024}{x^2} \quad \textbf{(1/2)}$$

Let C denotes the cost of the box.

$$C = 2x^2 \times 5 + 4xy \times 2.50 \quad \textbf{(1)}$$

$$= 10x^2 + 10xy = 10x(x + y)$$

$$= 10x\left(x + \frac{1024}{x^2}\right)$$

$$= 10x^2 + \frac{10240}{x} \quad \ldots(i) \ \textbf{(1/2)}$$

On differentiating both sides w.r.t. x, we get

$$\frac{dC}{dx} = 20x + 10240(-x)^{-2}$$

$$= 20x - \frac{10240}{x^2} \quad \ldots(ii)$$

Now, $\quad \frac{dC}{dx} = 0$

$$\Rightarrow \quad 20x = \frac{10240}{x^2}$$

$$\Rightarrow \quad 20x^3 = 10240$$

$$\Rightarrow \quad x^3 = 512 = 8^3 \Rightarrow x = 8 \quad \textbf{(1)}$$

Again, on differentiating Eq. (ii) w.r.t. x, we get

$$\frac{d^2C}{dx^2} = 20 - 10240(-2) \cdot \frac{1}{x^3}$$

$$= 20 + \frac{20480}{x^3} > 0$$

$$\left(\frac{d^2C}{dx^2}\right)_{x=8} = 20 + \frac{20480}{512} = 60 > 0 \quad \textbf{(1)}$$

For $x = 8$, cost is minimum and the corresponding least cost of the box

$$C(8) = 10 \cdot 8^2 + \frac{10240}{8} = 640 + 1280 = 1920$$

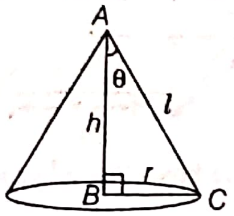
[using Eq. (i)]

Hence, least cost of the box is ₹ 1920. **(1)**

12. Let θ be the semi-vertical angle of the cone.

It is clear that $\theta \in \left(0, \frac{\pi}{2}\right)$

Let r, h and l be the radius, height and the slant height of the cone, respectively.



(1)

Since, slant height of the cone is given, so consider it as constant.

Now, in ΔABC, $r = l\sin\theta$ and $h = l\cos\theta$

Let V be the volume of the cone.

Then, $V = \frac{\pi}{3} r^2 h$

$\Rightarrow \quad V = \frac{1}{3}\pi(l^2\sin^2\theta)(l\cos\theta)$

$\Rightarrow \quad V = \frac{1}{3}\pi l^3 \sin^2\theta\cos\theta$ (1)

On differentiating both sides w.r.t. θ two times, we get

$$\frac{dV}{d\theta} = \frac{l^3\pi}{3}[\sin^2\theta(-\sin\theta) + \cos\theta(2\sin\theta\cos\theta)]$$

$$= \frac{l^3\pi}{3}(-\sin^3\theta + 2\sin\theta\cos^2\theta)$$

and $\frac{d^2V}{d\theta^2} = \frac{l^3\pi}{3}(-3\sin^2\theta\cos\theta + 2\cos^3\theta - 4\sin^2\theta\cos\theta)$

$\Rightarrow \quad \frac{d^2V}{d\theta^2} = \frac{l^3\pi}{3}(2\cos^3\theta - 7\sin^2\theta\cos\theta)$ (1)

For maxima or minima, put $\frac{dV}{d\theta} = 0$

$\Rightarrow \quad \sin^3\theta = 2\sin\theta\cos^2\theta$

$\Rightarrow \quad \tan^2\theta = 2$

$\Rightarrow \quad \tan\theta = \sqrt{2} \Rightarrow \theta = \tan^{-1}\sqrt{2}$ (1)

Now, when $\theta = \tan^{-1}\sqrt{2}$, then $\tan^2\theta = 2$

$\Rightarrow \quad \sin^2\theta = 2\cos^2\theta$

Now, we have

$$\frac{d^2V}{d\theta^2} = \frac{l^3\pi}{3}(2\cos^3\theta - 14\cos^3\theta)$$

$= -4\pi l^3\cos^3\theta < 0$, for $\theta \in \left(0, \frac{\pi}{2}\right)$ (1)

$\therefore V$ is maximum, when $\theta = \tan^{-1}\sqrt{2}$

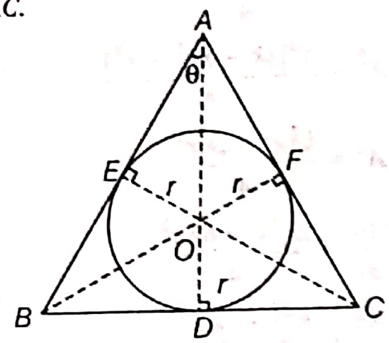
or $\quad \theta = \cos^{-1}\frac{1}{\sqrt{3}}$.

$$\left[\because \cos\theta = \frac{1}{\sqrt{1+\tan^2\theta}} = \frac{1}{\sqrt{1+2}} = \frac{1}{\sqrt{3}}\right]$$

Hence, for given slant height, the semi-vertical angle of the cone of maximum volume is $\cos^{-1}\frac{1}{\sqrt{3}}$. (1)

Hence proved.

13. Let ABC be the given isosceles triangle, with $AB = AC$.



Clearly, $OD \perp BC$, $OF \perp AC$, $OE \perp AB$

[$\because$ radius is perpendicular to the tangent at the point of contact]

and $BD = BE$, $CD = CF$, $AE = AF$

[$\because$ tangents from an external point to a circle are equal in length] (1)

Since, AD is an altitude of isosceles ΔABC.

Therefore, $BD = CD$

[$\because$ in an isosceles triangle, altitude from common vertex of equal sides bisect the third side]

$\Rightarrow \quad BD = BE = CF = CD$

[$\because BE = BD$ and $CD = CF$]

Now, perimeter (P) of $\Delta ABC = AB + BC + AC$

$= AE + BE + BD + DC + AF + FC$

$= (AE + AF) + (BE + BD + DC + FC)$

$= 2AE + 4BD$ ...(i) (1)

Consider ΔOEA, in this we have

$$AE = \frac{OE}{\tan\theta} = \frac{r}{\tan\theta} \text{ and } OA = \frac{r}{\sin\theta}$$

and in ΔADB we have $BD = AD\tan\theta$

$= (AO + OD)\tan\theta$

$= \left(\frac{r}{\sin\theta} + r\right)\tan\theta$ (1)

Now, $P = 2\cdot\frac{r}{\tan\theta} + 4\cdot\left(\frac{r}{\sin\theta} + r\right)\tan\theta$

[from Eq. (i)]

$\Rightarrow P(\theta) = r(2\cot\theta + 4\sec\theta + 4\tan\theta)$ ...(ii) **(1/2)**

On differentiating both sides w.r.t. θ, we get

$P'(\theta) = r(-2\text{cosec}^2\theta + 4\sec\theta\tan\theta + 4\sec^2\theta)$ ...(iii)

$$= r\left(\frac{-2}{\sin^2\theta} + \frac{4\sin\theta}{\cos^2\theta} + \frac{4}{\cos^2\theta}\right)$$

$$= r\left(\frac{-2\cos^2\theta + 4\sin^3\theta + 4\sin^2\theta}{\sin^2\theta\cos^2\theta}\right)$$ **(1/2)**

Now, put $P'(\theta) = 0$

$\Rightarrow -2\cos^2\theta + 4\sin^3\theta + 4\sin^2\theta = 0$

$\Rightarrow -2(1 - \sin^2\theta) + 4\sin^3\theta + 4\sin^2\theta = 0$

$\Rightarrow -2 + 2\sin^2\theta + 4\sin^3\theta + 4\sin^2\theta = 0$

$\Rightarrow 2\sin^3\theta + 3\sin^2\theta - 1 = 0$

$\Rightarrow (\sin\theta + 1)(2\sin^2\theta + \sin\theta - 1) = 0$

$\Rightarrow \sin\theta = -1$ or $2\sin^2\theta + \sin\theta - 1 = 0$

$\Rightarrow 2\sin^2\theta + \sin\theta - 1 = 0$

[$\because \sin\theta \neq -1$, as θ can't be more than 90°]

$\Rightarrow (2\sin\theta - 1)(\sin\theta + 1) = 0$

$\Rightarrow \sin\theta = \frac{1}{2}$ [$\because \sin\theta \neq -1$]

$\therefore \theta = \frac{\pi}{6}$ **(1)**

On differentiating both sides of Eq. (iii) w.r.t. θ, we get

$P''(\theta) = r(4\text{cosec}^2\theta\cot\theta + 4\sec^3\theta + 4\sec\theta\tan^2\theta + 8\sec^2\theta\tan\theta)$

$\Rightarrow P''\left(\frac{\pi}{6}\right) > 0$

Thus, $P(\theta)$ is minimum, when $\theta = \frac{\pi}{6}$ **(1/2)**

Now, from Eq. (i), least perimeter $= P\left(\frac{\pi}{6}\right)$

$$= r\left[2\cot\left(\frac{\pi}{6}\right) + 4\sec\left(\frac{\pi}{6}\right) + 4\tan\left(\frac{\pi}{6}\right)\right]$$

$$= r\left(2\sqrt{3} + 4\cdot\frac{2}{\sqrt{3}} + 4\cdot\frac{1}{\sqrt{3}}\right)$$

$$= r\left(\frac{6 + 8 + 4}{\sqrt{3}}\right) = \frac{18}{\sqrt{3}}r = 6\sqrt{3}r$$ **(1/2)**

Hence proved.

14. Let r be the radius of the sphere and dimensions of cuboid are x, $2x$ and $\frac{x}{3}$.

$\therefore 4\pi r^2 + 2\left[\frac{x}{3}\times x + x\times 2x + 2x\times\frac{x}{3}\right]$

$= k$ (constant) [given]

$\Rightarrow 4\pi r^2 + 6x^2 = k$

$\Rightarrow r^2 = \frac{k - 6x^2}{4\pi} \Rightarrow r = \sqrt{\frac{k - 6x^2}{4\pi}}$ ...(i) **(1)**

Sum of the volumes, $V = \frac{4}{3}\pi r^3 + \frac{x}{3}\times x\times 2x$

$$= \frac{4\pi r^3}{3} + \frac{2}{3}x^3$$ ...(ii)

$\Rightarrow V = \frac{4}{3}\pi\left(\frac{k - 6x^2}{4\pi}\right)^{\frac{3}{2}} + \frac{2}{3}x^3$ **(1)**

On differentiating both sides w.r.t. x, we get

$$\frac{dV}{dx} = \frac{4}{3}\pi\times\frac{3}{2}\left(\frac{k - 6x^2}{4\pi}\right)^{\frac{1}{2}}\left(\frac{-12x}{4\pi}\right) + \frac{2}{3}\times 3x^2$$

$$= 2\pi\sqrt{\frac{k - 6x^2}{4\pi}}\left(\frac{-3x}{\pi}\right) + 2x^2$$

$$= (-6x)\sqrt{\frac{k - 6x^2}{4\pi}} + 2x^2$$ **(1)**

For maxima or minima, put $\frac{dV}{dx} = 0$

$\Rightarrow (-6x)\sqrt{\frac{k - 6x^2}{4\pi}} + 2x^2 = 0$

$\Rightarrow 2x^2 = 6x\sqrt{\frac{k - 6x^2}{4\pi}} \Rightarrow x = 3\sqrt{\frac{k - 6x^2}{4\pi}}$

$\Rightarrow x = 3r$ [using Eq. (i)] **(1)**

Again, on differentiating $\frac{dV}{dx}$ w.r.t. x, we get

$$\frac{d^2V}{dx^2} = -6\frac{d}{dx}\left(x\sqrt{\frac{k - 6x^2}{4\pi}}\right) + 4x$$

$$= -6\left(\sqrt{\frac{k - 6x^2}{4\pi}} + x\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{\frac{k - 6x^2}{4\pi}}}\left(\frac{-12x}{4\pi}\right)\right) + 4x$$

$$= -6\left(r - \frac{3x^2}{2\pi r}\right) + 4x = -6r + \frac{9x^2}{\pi r} + 4x$$

Now, $\left(\frac{d^2r}{dx^2}\right)_{x=3r} = -6r + \frac{9\times 9r^2}{\pi r} + 12r = 6r + \frac{18r}{\pi} > 0$

Hence, V is minimum when x is equal to three times the radius of the sphere. **(1)**

Hence proved.

Now, on putting $r = \frac{x}{3}$ in Eq. (ii), we get

$$V_{\min} = \frac{4\pi}{3}\left(\frac{x}{3}\right)^3 + \frac{2}{3}x^3 = \frac{4\pi}{81}x^3 + \frac{2}{3}x^3$$

$$= \frac{2}{3}x^2\left(\frac{2\pi}{27}+1\right) = \frac{2}{3}x^3\left(\frac{44}{189}+1\right)$$

$$= \frac{2}{3}x^3\left(\frac{233}{189}\right) = \frac{466}{567}x^3 \quad \textbf{(1)}$$

15. We have, $f(x) = \sin x - \cos x,\ 0 < x < 2\pi$

On differentiating both sides w.r.t. x, we get

$$f'(x) = \cos x + \sin x \quad \dots \text{(i)}$$

For local maximum and local minimum,
Put $f'(x) = 0$,

i.e. $\cos x + \sin x = 0 \Rightarrow \cos x = -\sin x$ **(1)**

$\Rightarrow \tan x = -1 \Rightarrow x = \pi - \frac{\pi}{4}$ or $2\pi - \frac{\pi}{4}$

$\Rightarrow x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$ **(1)**

Again, on differentiating both sides of Eq. (i) w.r.t. x, we get

$$f''(x) = -\sin x + \cos x$$

When $x = \frac{3\pi}{4}$, then

$$f''\left(\frac{3\pi}{4}\right) = -\sin\frac{3\pi}{4} + \cos\frac{3\pi}{4} \quad \textbf{(1)}$$

$$= -\sin\left(\pi - \frac{\pi}{4}\right) + \cos\left(\pi - \frac{\pi}{4}\right)$$

$$= -\sin\frac{\pi}{4} - \cos\frac{\pi}{4} < 0$$

When $x = \frac{7\pi}{4}$, then $f''\left(\frac{7\pi}{4}\right) = -\sin\frac{7\pi}{4} + \cos\frac{7\pi}{4}$

$$= -\sin\left(2\pi - \frac{\pi}{4}\right) + \cos\left(2\pi - \frac{\pi}{4}\right)$$

$$= \sin\frac{\pi}{4} + \cos\frac{\pi}{4} > 0 \quad \textbf{(1)}$$

Thus, $x = \frac{3\pi}{4}$ is a point of local maxima and $x = \frac{7\pi}{4}$ is a point of local minima.

Now, the local maximum value,

$$f\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{4} - \cos\frac{3\pi}{4}$$

$$= \sin\left(\pi - \frac{\pi}{4}\right) - \cos\left(\pi - \frac{\pi}{4}\right)$$

$$= \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2} \quad \textbf{(1)}$$

and the local minimum value,

$$f\left(\frac{7\pi}{4}\right) = \sin\frac{7\pi}{4} - \cos\frac{7\pi}{4}$$

$$= \sin\left(2\pi - \frac{\pi}{4}\right) - \cos\left(2\pi - \frac{\pi}{4}\right)$$

$$= -\sin\frac{\pi}{4} - \cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}$$

$$= -\frac{2}{\sqrt{2}} = -\sqrt{2} \quad \textbf{(1)}$$

16. Let $f(x) = ax + by$, whose minimum value is required.

Then, $f(x) = ax + \frac{bc^2}{x}$ $\left[\text{given, } xy = c^2 \Rightarrow y = \frac{c^2}{x}\right]$

On differentiating both sides w.r.t. x, we get

$$f'(x) = a - \frac{bc^2}{x^2} \quad \textbf{(1)}$$

For maxima or minima, put $f'(x) = 0$

$\Rightarrow a - \frac{bc^2}{x^2} = 0$

$\Rightarrow a = \frac{bc^2}{x^2} \Rightarrow x^2 = \frac{bc^2}{a} \Rightarrow x = \pm\sqrt{\frac{b}{a}}\,c$ **(1)**

Now, $f''(x) = 0 + \frac{2bc^2}{x^3}$

At $x = \sqrt{\frac{b}{a}}\,c$, $f''(x) = \frac{2bc^2}{\left(\sqrt{\frac{b}{a}}\,c\right)^3} = +\text{ve}$

Hence, $f(x)$ has minimum value at $x = \sqrt{\frac{b}{a}}\,c$. **(1)**

At $x = -\sqrt{\frac{b}{a}}\,c$, $f''(x) = \frac{2bc^2}{\left(-\sqrt{\frac{b}{a}}\,c\right)^3} = -\text{ve}$

Hence, $f(x)$ has maximum value at $x = -\sqrt{\frac{b}{a}}\,c$. **(1)**

When $x = \sqrt{\frac{b}{a}}\,c$, then

$$y = \frac{c^2}{x} = \frac{c^2}{\left(\sqrt{\frac{b}{a}}\,c\right)} = \sqrt{\frac{a}{b}}\,c \quad \textbf{(1)}$$

∴ Minimum value of $f(x) = a\sqrt{\frac{b}{a}}\cdot c + b\sqrt{\frac{a}{b}}\cdot c$

$$= \sqrt{ab}\cdot c + \sqrt{ab}\cdot c$$
$$= 2\sqrt{ab}\cdot c$$ **(1)**

17. Given equation of curve is $y = x^2 + 7x + 2$ and equation of straight line is $y = 3x - 3$.

Let $P(x, y)$ be any point on the parabola $y = x^2 + 7x + 2$.

Let D be the distance of point P from straight line, then

$$D = \frac{|3x - y - 3|}{\sqrt{3^2 + (-1)^2}} = \frac{|3x - y - 3|}{\sqrt{9+1}}$$ **(1)**

$$= \frac{|3x - y - 3|}{\sqrt{10}} = \frac{|3x - (x^2 + 7x + 2) - 3|}{\sqrt{10}}$$ **(1)**

$$= \frac{|-(x^2 + 4x + 5)|}{\sqrt{10}} = \frac{x^2 + 2\cdot 2x + 2^2 + 1}{\sqrt{10}}$$

$$\Rightarrow D = \frac{(x+2)^2 + 1}{\sqrt{10}}$$ ...(i) **(1)**

On differentiating both sides of Eq. (i) w.r.t. x, we get

$$\frac{dD}{dx} = \frac{2(x+2) + 0}{\sqrt{10}}$$

For extremum value of D, put $\frac{dD}{dx} = 0$

$$\Rightarrow 2(x+2) = 0$$
$$\Rightarrow x = -2$$ **(1)**

Now, $$\frac{d^2D}{dx^2} = \frac{2}{\sqrt{10}} > 0$$

Thus, D is minimum when $x = -2$ **(1)**

Now, $y = x^2 + 7x + 2 = (-2)^2 + 7(-2) + 2$

$$= 4 - 14 + 2 = -8$$

Hence, point $(-2, -8)$ is on the parabola, which is closest to the given straight line **(1)**

18. Let P be a point on the hypotenuse AC of right angled ΔABC. Such that $PL \perp AB$ and $PL = a$ and $PM \perp BC$ and $PM = b$. [say]

Let $\angle APL = \angle ACB = \theta$ **(1)**

Then, $AP = a\sec\theta$, $PC = b\,\text{cosec}\,\theta$

Let l be the length of the hypotenuse, then

$$l = AP + PC$$
$$\Rightarrow l = a\sec\theta + b\,\text{cosec}\,\theta,\ 0 < \theta < \frac{\pi}{2}$$

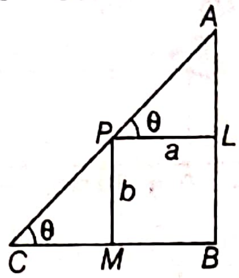
On differentiating both sides w.r.t. θ, we get

$$\frac{dl}{d\theta} = a\sec\theta\tan\theta - b\,\text{cosec}\,\theta\cot\theta$$ ...(i)

For maxima or minima, put $\frac{dl}{d\theta} = 0$ **(1)**

$$\Rightarrow a\sec\theta\tan\theta = b\,\text{cosec}\,\theta\cot\theta$$

$$\Rightarrow \frac{a\sin\theta}{\cos^2\theta} = \frac{b\cos\theta}{\sin^2\theta} \Rightarrow \tan\theta = \left(\frac{b}{a}\right)^{1/3}$$



(1)

Again, on differentiating both sides of Eq. (i) w.r.t. θ, we get

$$\frac{d^2l}{d\theta^2} = a(\sec\theta\times\sec^2\theta + \tan\theta\times\sec\theta\tan\theta) - b[\text{cosec}\,\theta(-\text{cosec}^2\theta) + \cot\theta(-\text{cosec}\,\theta\cot\theta)]$$

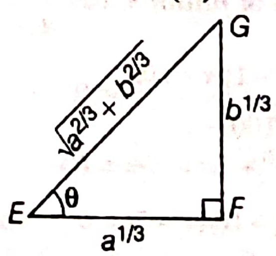
$$= a\sec\theta(\sec^2\theta + \tan^2\theta) + b\,\text{cosec}\,\theta(\text{cosec}^2\theta + \cot^2\theta)$$

For $0 < \theta < \frac{\pi}{2}$, all trigonometric ratios are positive. **(1)**

Also, $a > 0$ and $b > 0$.

∴ $\frac{d^2l}{d\theta^2}$ is positive. **(1/2)**

$\Rightarrow l$ is least when $\tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$



∴ Least value of,

$$l = a\sec\theta + b\,\text{cosec}\,\theta$$

$$= a\frac{\sqrt{a^{2/3} + b^{2/3}}}{a^{1/3}} + b\frac{\sqrt{a^{2/3} + b^{2/3}}}{b^{1/3}}$$

$$= \sqrt{a^{2/3} + b^{2/3}}(a^{2/3} + b^{2/3}) = (a^{2/3} + b^{2/3})^{3/2}$$ **(1½)**

$$\left[\because \text{in } \Delta EFG, \tan\theta = \frac{b^{1/3}}{a^{1/3}}, \sec\theta = \frac{\sqrt{a^{2/3} + b^{2/3}}}{a^{1/3}} \text{ and } \text{cosec}\,\theta = \frac{\sqrt{a^{2/3} + b^{2/3}}}{b^{1/3}}\right]$$

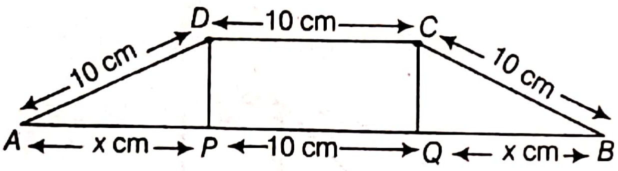
Hence proved.

19. Let $ABCD$ be the given trapezium in which $AD = BC = CD = 10$ cm.

Let $\quad AP = x$ cm

$\because \quad \Delta APD \cong \Delta BQC$

$\therefore \quad QB = x$ cm **(1)**



In ΔAPD,

$DP = \sqrt{10^2 - x^2}$ [by Pythagoras theorem]

Now, area of trapezium,

$$A = \frac{1}{2} \times (\text{sum of parallel sides}) \times \text{height}$$

$$= \frac{1}{2} \times (2x + 10 + 10) \times \sqrt{100 - x^2}$$

$$= (x + 10)\sqrt{100 - x^2} \quad \text{...(i)} \ \textbf{(1)}$$

On differentiating both sides of Eq. (i) w.r.t. x, we get

$$\frac{dA}{dx} = (x + 10)\frac{(-2x)}{2\sqrt{100 - x^2}} + \sqrt{100 - x^2}$$

$$= \frac{-x^2 - 10x + 100 - x^2}{\sqrt{100 - x^2}}$$

$$= \frac{-2x^2 - 10x + 100}{\sqrt{100 - x^2}} \quad \text{...(ii)} \ \textbf{(1)}$$

For maxima or minima, put $\frac{dA}{dx} = 0$

$$\Rightarrow \quad \frac{-2x^2 - 10x + 100}{\sqrt{100 - x^2}} = 0$$

$$\Rightarrow \quad -2(x^2 + 5x - 50) = 0$$

$$\Rightarrow \quad -2(x + 10)(x - 5) = 0$$

$$\Rightarrow \quad x = 5 \text{ or } -10 \quad \textbf{(1)}$$

Since, x represents distance, so it cannot be negative.

Therefore, we take $x = 5$.

On differentiating both sides of Eq. (ii) w.r.t. x, we get

$$\frac{d^2A}{dx^2} = \frac{\left[\begin{array}{l}\sqrt{100 - x^2}(-4x - 10) \\ -(-2x^2 - 10x + 100)\left(\frac{-2x}{2\sqrt{100 - x^2}}\right)\end{array}\right]}{(\sqrt{100 - x^2})^2}$$

[by using quotient rule of derivative]

$$= \frac{\left[\begin{array}{l}(100 - x^2)(-4x - 10) \\ \quad -(-2x^2 - 10x + 100)(-x)\end{array}\right]}{(100 - x^2)^{3/2}}$$

$$= \frac{\left[\begin{array}{l}-400x - 1000 + 4x^3 + 10x^2 \\ \quad +(-2x^3 - 10x^2 + 100x)\end{array}\right]}{(100 - x^2)^{3/2}}$$

$$= \frac{2x^3 - 300x - 1000}{(100 - x^2)^{3/2}} \quad \textbf{(1)}$$

At $x = 5$, $\frac{d^2A}{dx^2} = \frac{2(5)^3 - 300(5) - 1000}{[100 - (5)^2]^{3/2}}$

$$= \frac{250 - 1500 - 1000}{(100 - 25)^{3/2}} = \frac{-2250}{75\sqrt{75}} < 0$$

Thus, area of trapezium is maximum at $x = 5$ and maximum area is

$A_{max} = (5 + 10)\sqrt{100 - (5)^2}$ [put $x = 5$ in Eq. (i)]

$$= 15\sqrt{100 - 25} = 15\sqrt{75} = 75\sqrt{3} \text{ cm}^2 \quad \textbf{(1)}$$

20. Let $P(x, y)$ be any point on $y^2 = 4ax$. ...(i)

Then, distance between (x, y) and $(11a, 0)$ is given by

$$D = \sqrt{(x - 11a)^2 + (y - 0)^2} = \sqrt{(x - 11a)^2 + y^2}$$

$$= \sqrt{(x - 11a)^2 + 4ax} \quad \text{[from Eq. (i)]} \ \textbf{(1)}$$

On differentiating both sides w.r.t. x, we get

$$\frac{dD}{dx} = \frac{[2(x - 11a) + 4a]}{2\sqrt{(x - 11a)^2 + 4ax}} \quad \textbf{(1)}$$

$$\Rightarrow \quad \frac{dD}{dx} = \frac{2x - 22a + 4a}{2\sqrt{(x - 11a)^2 + 4ax}}$$

$$\Rightarrow \quad \frac{dD}{dx} = \frac{x - 9a}{\sqrt{(x - 11a^2) + 4ax}}$$

Put $\frac{dD}{dx} = 0 \Rightarrow x - 9a = 0 \Rightarrow x = 9a$ **(1)**

Now, $\quad \frac{d^2D}{dx^2} = \frac{d}{dx}\left(\frac{dD}{dx}\right)$ **(1)**

$$= \frac{d}{dx}\left(\frac{x - 9a}{\sqrt{(x - 11a)^2 + 4ax}}\right)$$

$$= \frac{\sqrt{(x - 11a)^2 + 4ax} - (x - 9a) \cdot \frac{1 \cdot [2(x - 11a) + 4a]}{2\sqrt{(x - 11a)^2 + 4ax}}}{(x - 11a)^2 + 4ax}$$

$$\therefore \left(\frac{d^2D}{dx^2}\right)_{x=9} = \frac{\sqrt{(9a-11a)^2+4a\cdot 9a}-0}{(9a-11a)^2+4a\times 9a}$$

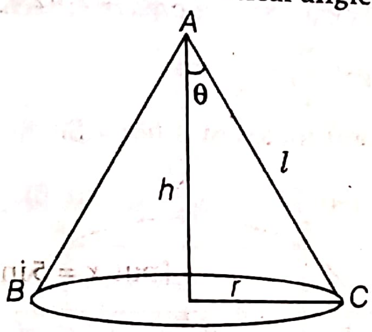
$$= \frac{\sqrt{4a^2+36a^2}}{4a^2+36a^2} = \frac{1}{\sqrt{40a^2}} > 0 \quad \textbf{(1)}$$

So, at $(x = 9a)$, D is minimum.

Now, $y^2 = 36a^2 \Rightarrow y = \pm 6a$

Hence, required points are $(9a, 6a)$ and $(9a, -6a)$. **(1)**

21. Let r be the radius of the base, h be the height, V be the volume, S be the surface area of the cone ABC and θ be the semi-vertical angle.



(1)

Then, $V = \frac{1}{3}\pi r^2 h \Rightarrow 3V = \pi r^2 h$

$\Rightarrow \quad 9V^2 = \pi^2 r^4 h^2$ [on squaring both sides]

$\Rightarrow \quad h^2 = \frac{9V^2}{\pi^2 r^4}$ ...(i) **(1)**

and curved surface area, $S = \pi r l$

$\Rightarrow \quad S = \pi r\sqrt{r^2+h^2}$ $[\because l = \sqrt{h^2+r^2}]$

$\Rightarrow \quad S^2 = \pi^2 r^2 (r^2+h^2)$ [on squaring both sides]

$\Rightarrow \quad S^2 = \pi^2 r^2\left(\frac{9V^2}{\pi^2 r^4} + r^2\right)$ [from Eq. (i)]

$\Rightarrow \quad S^2 = \frac{9V^2}{r^2} + \pi^2 r^4$ ...(ii)

When S is least, then S^2 is also least. **(1)**

Now, $\frac{d}{dr}(S^2) = -\frac{18V^2}{r^3} + 4\pi^2 r^3$ ...(iii)

For maxima or minima, put $\frac{d}{dr}(S^2) = 0$

$\Rightarrow \quad -\frac{18V^2}{r^3} + 4\pi^2 r^3 = 0$

$\Rightarrow \quad 18V^2 = 4\pi^2 r^6$

$\Rightarrow \quad 9V^2 = 2\pi^2 r^6$ ...(iv) **(1)**

Again, on differentiating Eq. (iii) w.r.t. r, we get

$$\frac{d^2}{dr^2}(S^2) = \frac{54V^2}{r^4} + 12\pi^2 r^2 > 0$$

At $r = \left(\frac{9V^2}{2\pi^2}\right)^{1/6}$, $\frac{d^2}{dr^2}(S^2) > 0$

So, S^2 or S is minimum, when

$V^2 = 2\pi^2 r^6/9$ **(1)**

On putting $V^2 = 2\pi^2 r^6/9$ in Eq. (i), we get

$2\pi^2 r^6 = \pi^2 r^4 h^2$

$\Rightarrow \quad 2r^2 = h^2 \Rightarrow h = \sqrt{2}\, r \Rightarrow \frac{h}{r} = \sqrt{2}$

$\Rightarrow \quad \cot\theta = \sqrt{2}$ $\left[\text{from the figure, } \cot\theta = \frac{h}{r}\right]$

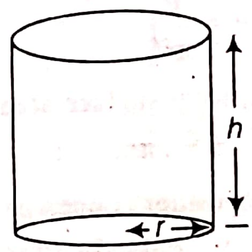
$\therefore \quad \theta = \cot^{-1}\sqrt{2}$

Hence, the semi-vertical angle of the right circular cone of given volume and least curved surface area is $\cot^{-1}\sqrt{2}$. **(1)**

Hence proved

NOTE If square of any area is maximum (or minimum), then area is also maximum (or minimum).

22. Let r cm be the radius of base and h cm be the height of the cylindrical can. Let its volume be V and S be its total surface area.



(1)

Then, $V = 128\pi \text{ cm}^3$ [givn]

$\Rightarrow \quad \pi r^2 h = 128\pi \Rightarrow h = \frac{128}{r^2}$ ...(i)

Now, surface area of cylindrical can,

$S = 2\pi r^2 + 2\pi r h$ ..(ii)

$\Rightarrow \quad S = 2\pi r^2 + 2\pi r\left(\frac{128}{r^2}\right)$ [using Eq.(i)] **(1)**

$\Rightarrow \quad S = 2\pi r^2 + \frac{256\pi}{r}$ ..(iii)

On differentiating both sides of Eq. (iii) w.r.t r, we get

$$\frac{dS}{dr} = 4\pi r - \frac{256\pi}{r^2} \quad \text{...(v)} \ \textbf{(1)}$$

For maxima or minima, put $\frac{dS}{dr} = 0$

$\Rightarrow \quad 4\pi r = \frac{256\pi}{r^2} \Rightarrow r^3 = \frac{256}{4} \Rightarrow r^3 = 64$

Taking cube root on both sides, we get

$$r = (64)^{1/3} \Rightarrow r = 4 \text{ cm} \quad (1)$$

Again, on differentiating Eq. (iv) w.r.t. r, we get

$$\frac{d^2S}{dr^2} = 4\pi + \frac{512\pi}{r^3}$$

At $r = 4$,

$$\frac{d^2S}{dr^2} = \frac{512\pi}{64} + 4\pi = 8\pi + 4\pi = 12\pi > 0 \quad (1)$$

Thus, $\frac{d^2S}{dr^2} > 0$ at $r = 4$, so the surface area is minimum, when the radius of cylinder is 4 cm. On putting the value of r in Eq. (i), we get

$$h = \frac{128}{(4)^2} = \frac{128}{16} = 8 \text{ cm}$$

Hence, for the minimum surface area of can, the dimensions of the cylindrical can are $r = 4$ cm and $h = 8$ cm. (1)

23. Let r be the radius, h be the height, V be the volume and S be the total surface area of a right circular cylinder which is open at the top. Now, given that $V = \pi r^2 h$

$$\Rightarrow \quad h = \frac{V}{\pi r^2} \quad \text{...(i) (1)}$$

We know that, total surface area S is given by

$$S = 2\pi rh + \pi r^2$$

[∵ cylinder is open at the top, therefore S = curved surface area of cylinder + area of base]

$$\Rightarrow \quad S = 2\pi r\left(\frac{V}{\pi r^2}\right) + \pi r^2$$

$$\left[\text{put } h = \frac{V}{\pi r^2}, \text{ from Eq. (i)}\right]$$

$$\Rightarrow \quad S = \frac{2V}{r} + \pi r^2 \quad (1)$$

On differentiating both sides w.r.t. r, we get

$$\frac{dS}{dr} = -\frac{2V}{r^2} + 2\pi r$$

For maxima or minima, put $\frac{dV}{dr} = 0$

$$\Rightarrow \quad -\frac{2V}{r^2} + 2\pi r = 0 \Rightarrow V = \pi r^3$$

$$\Rightarrow \quad \pi r^2 h = \pi r^3 \quad [\because V = \pi r^2 h]$$

$$\Rightarrow \quad h = r \quad (1)$$

Also, $\frac{d^2S}{dr^2} = \frac{d}{dr}\left(\frac{dS}{dr}\right) = \frac{d}{dr}\left(\frac{-2V}{r^2} + 2\pi r\right)$

$$\Rightarrow \quad \frac{d^2S}{dr^2} = \frac{4V}{r^3} + 2\pi \quad (1)$$

On putting $r = h$, we get

$$\left[\frac{d^2S}{dr^2}\right]_{r=h} = \frac{4V}{h^3} + 2\pi > 0, \text{ as } h > 0. \quad (1)$$

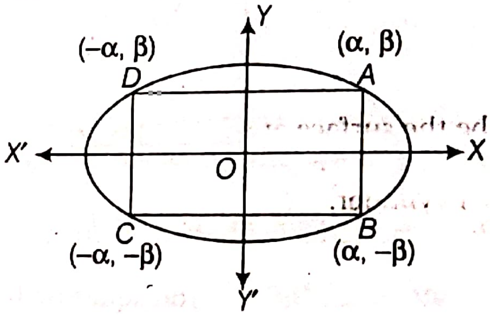
Then, $\frac{d^2S}{dr^2} > 0 \Rightarrow S$ is minimum.

Hence, S is minimum, when $h = r$, i.e. when height of cylinder is equal to radius of the base. (1)

Hence proved.

24. Let $ABCD$ be a rectangle having area A inscribed in an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. ...(i)

Let the coordinates of A be (α, β).



Then, the coordinates of $B = (\alpha, -\beta)$

$C = (-\alpha, -\beta)$

$D = (-\alpha, \beta)$ (1)

∴ Area of rectangle, A = Length × Breadth

$= 2\alpha \times 2\beta \Rightarrow A = 4\alpha\beta$

$$\Rightarrow \quad A = 4\alpha \cdot \sqrt{b^2\left(1 - \frac{\alpha^2}{a^2}\right)} \quad (1)$$

$$\left[\begin{array}{l}\because (\alpha, \beta) \text{ lies on ellipse} \\ \therefore \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} = 1, \text{ i.e. } \beta = \sqrt{b^2\left(\frac{a^2 - \alpha^2}{a^2}\right)}\end{array}\right]$$

$$\Rightarrow \quad A^2 = 16\alpha^2\left\{b^2\left(\frac{a^2 - \alpha^2}{a^2}\right)\right\}$$

[on squaring both sides]

$$\Rightarrow \quad A^2 = \frac{16b^2}{a^2}(a^2\alpha^2 - \alpha^4) \quad (1)$$

On differentiating both sides w.r.t. α, we get

$$\frac{d(A^2)}{d\alpha} = \frac{16b^2}{a^2}(2a^2\alpha - 4\alpha^3)$$

For maxima or minima, put $\frac{dA^2}{d\alpha} = 0$

$$\Rightarrow \quad 2a^2\alpha - 4\alpha^3 = 0$$

$$\Rightarrow \quad 2\alpha(a^2 - 2\alpha^2) = 0$$

$$\Rightarrow \quad \alpha = 0, \ \alpha = \frac{a}{\sqrt{2}}$$ (1)

Again, $$\frac{d^2(A^2)}{d\alpha^2} = \frac{16b^2}{a^2}(2a^2 - 12\alpha^2)$$

$$\therefore \left(\frac{d^2A^2}{d\alpha^2}\right)_{\alpha=\frac{a}{\sqrt{2}}} = \frac{16b^2}{a^2}\left(2a^2 - 12\times\frac{a^2}{2}\right)$$

$$= -64b^2 < 0$$ (1)

$\Rightarrow$ For $\alpha = \frac{a}{\sqrt{2}}$, A^2, i.e. A is maximum.

Then, from Eq. (i), we get

$$\beta = \sqrt{\frac{b^2\left(a^2 - \frac{a^2}{2}\right)}{a^2}} = \frac{b}{\sqrt{2}}$$

$\therefore$ Greatest area $= 4\cdot\alpha\beta = 4\cdot\frac{a}{\sqrt{2}}\cdot\frac{b}{\sqrt{2}} = 2ab$ (1)

25. Let S be the surface area, V be the volume, h be the height and r be the radius of base of the right circular cylinder.

We know that, Surface area of right circular cylinder,

$$S = 2\pi r^2 + 2\pi rh \quad \text{...(i)}$$

$$\Rightarrow \quad h = \frac{S - 2\pi r^2}{2\pi r} \quad \text{...(ii)}$$ (1)

Also, volume of right circular cylinder is given by

$$V = \pi r^2 h$$

$$\Rightarrow \quad V = \pi r^2\left(\frac{S - 2\pi r^2}{2\pi r}\right) \quad \text{[from Eq. (ii)]}$$

$$\Rightarrow \quad V = \frac{rS - 2\pi r^3}{2}$$ (1)

On differentiating both sides w.r.t. r, we get

$$\frac{dV}{dr} = \frac{S - 6\pi r^2}{2}$$ (1)

For maxima or minima, put $\frac{dV}{dr} = 0$

$$\Rightarrow \quad \frac{S - 6\pi r^2}{2} = 0$$

$$\Rightarrow \quad S = 6\pi r^2$$ (1)

From Eq. (ii), we get

$$h = \frac{6\pi r^2 - 2\pi r^2}{2\pi r}$$

$$\Rightarrow \quad h = 2r$$

$\therefore$ Height = Diameter of the base (1)

Also, $$\frac{d^2V}{dr^2} = \frac{d}{dr}\left(\frac{dV}{dr}\right) = \frac{d}{dr}\left(\frac{S - 6\pi r^2}{2}\right)$$

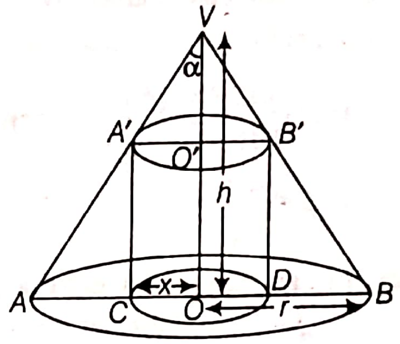
$$= -6\pi r < 0$$

$\therefore V$ is maximum.

Hence, V is maximum at $h = 2r$. (1)

Hence proved.

26. Let VAB be the cone of base radius r, height h and radius of base of the inscribed cylinder be x.



(1)

Now, we observe that

$$\Delta VOB \sim \Delta B'DB \quad \Rightarrow \quad \frac{VO}{B'D} = \frac{OB}{DB}$$

[$\because$ if the triangles are similar, then their sides are proportional]

$$\Rightarrow \quad \frac{h}{B'D} = \frac{r}{r - x}$$

$$\Rightarrow \quad B'D = \frac{h(r - x)}{r}$$ (1)

Let C be the curved surface area of cylinder. Then,

$$C = 2\pi\,(OC)\,(B'D)$$

$$\Rightarrow \quad C = \frac{2\pi xh(r - x)}{r} = \frac{2\pi h}{r}(rx - x^2)$$ (1)

On differentiating both sides w.r.t. x, we get

$$\frac{dC}{dx} = \frac{2\pi h}{r}(r - 2x)$$

For maxima or minima, put $\frac{dC}{dx} = 0$

$$\Rightarrow \quad \frac{2\pi h}{r}(r - 2x) = 0$$

$$\Rightarrow \quad r - 2x = 0 \ \Rightarrow r = 2x$$

$$\therefore \quad x = \frac{r}{2}$$ (1)

Hence, radius of cylinder is half of that of cone.

Also, $$\frac{d^2C}{dx^2} = \frac{d}{dx}\left[\frac{2\pi h(r - 2x)}{r}\right]$$

$$= \frac{2\pi h}{r}(-2) = \frac{-4\pi h}{r} < 0 \text{ as } h, r > 0$$ (1)

$\therefore C$ is maximum or greatest.

Hence, C is greatest at $x = \frac{r}{2}$. **Hence proved.** (1)

27. Let the dimensions of the box be x and y. Also, let V denotes its volume and S denotes its total surface area.

Now, $S = x^2 + 4xy$ $\left[\because S = \text{area of square base} + \text{area of the four walls}\right]$

Given, $x^2 + 4xy = C^2$

$$\Rightarrow \quad y = \frac{C^2 - x^2}{4x} \quad \ldots(\text{i}) \ (1)$$

Also, volume of the box is given by

$$V = x^2 y$$

$$\Rightarrow \quad V = x^2\left(\frac{C^2 - x^2}{4x}\right) \quad [\text{from Eq. (i)}]$$

$$\Rightarrow \quad V = \frac{xC^2 - x^3}{4} \quad (1)$$

On differentiating both sides w.r.t. x, we get

$$\frac{dV}{dx} = \frac{C^2 - 3x^2}{4} \quad (1)$$

For maxima or minima, put $dV/dx = 0$

$$\Rightarrow \quad \frac{C^2 - 3x^2}{4} = 0$$

$$\Rightarrow \quad C^2 = 3x^2$$

$$\therefore \quad x = C/\sqrt{3} \quad (1)$$

Also, $$\frac{d^2V}{dx^2} = \frac{d}{dx}\left(\frac{dV}{dx}\right) = \frac{d}{dx}\left(\frac{C^2 - 3x^2}{4}\right)$$

$$= \frac{-6x}{4} = \frac{-3x}{2}$$

$$\therefore \quad \left.\frac{d^2V}{dx^2}\right|_{\text{at } x = C/\sqrt{3}} < 0$$

$\Rightarrow$ V is maximum. (1)

Now, maximum volume at $x = \frac{C}{\sqrt{3}}$ is

$$V = \frac{xC^2 - x^3}{4}$$

$$= \frac{1}{4}\left[\frac{C}{\sqrt{3}} \cdot C^2 - \left(\frac{C}{\sqrt{3}}\right)^3\right] \quad \left[\text{put } x = \frac{C}{\sqrt{3}}\right]$$

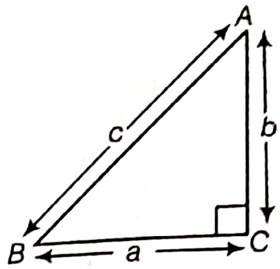
$$= \frac{1}{4}\left[\frac{C^3}{\sqrt{3}} - \frac{C^3}{3\sqrt{3}}\right] = \frac{1}{4}\left[\frac{3C^3 - C^3}{3\sqrt{3}}\right]$$

$$= \frac{1}{4} \times \frac{2C^3}{3\sqrt{3}} = \frac{C^3}{6\sqrt{3}}$$

Hence, the maximum volume of box is $\frac{C^3}{6\sqrt{3}}$ cu units. (1)

Hence proved.

28. Let a and b be the sides of right angled triangle and c be the hypotenuse.



(1/2)

From ΔABC, we have

$$c^2 = a^2 + b^2$$

Area of ΔABC, $(A) = \frac{1}{2} a \cdot b = \frac{1}{2} a\sqrt{c^2 - a^2}$

$[\because b = \sqrt{c^2 - a^2}]$ (1)

On differentiating both sides w.r.t. a, we get

$$\frac{dA}{da} = \frac{1}{2} \cdot 1 \cdot \sqrt{c^2 - a^2} + \frac{1}{2} \cdot a \cdot \frac{1}{2} \cdot \frac{(-2a)}{\sqrt{c^2 - a^2}}$$

$$= \frac{1}{2}\left(\sqrt{c^2 - a^2} - \frac{a^2}{\sqrt{c^2 - a^2}}\right) \quad (1)$$

For maxima or minima, put $\frac{dA}{da} = 0$

$$\Rightarrow \quad \frac{1}{2}\left(\sqrt{c^2 - a^2} - \frac{a^2}{\sqrt{c^2 - a^2}}\right) = 0$$

$$\Rightarrow \quad c^2 - a^2 - a^2 = 0$$

$$\Rightarrow \quad c^2 = 2a^2 \Rightarrow a = \frac{c}{\sqrt{2}}$$

Now, $$\frac{d^2A}{da^2} = \frac{1}{2}\left[\frac{-a}{\sqrt{c^2 - a^2}} - \frac{a^3}{(c^2 - a^2)^{3/2}}\right]$$

$$= -\frac{1}{2} a\left[\frac{c^2 - a^2 + a^2}{(c^2 - a^2)^{3/2}}\right] \quad (1)$$

$$= -\frac{1}{2} \frac{c^2 a}{(c^2 - a^2)^{3/2}} < 0 \quad (1½)$$

$\therefore$ Area of ΔABC is maximum and

$$b = \sqrt{c^2 - a^2} = \sqrt{2a^2 - a^2} = a \quad (1)$$

Hence, the triangle is isosceles. **Hence proved.**

29. Let C denotes the curved surface area, r be the radius of base, h be the height and V be the volume of right circular cone.

We know that, volume of cone is given by

$$V = \frac{1}{3}\pi r^2 h \Rightarrow h = \frac{3V}{\pi r^2} \quad \ldots(\text{i}) \ (1)$$

Also, the curved surface area of cone is given by $C = \pi r l$, where $l = \sqrt{r^2 + h^2}$ is the slant height of cone.

$$\therefore \quad C = \pi r \sqrt{r^2 + h^2}$$

On squaring both sides, we get

$$C^2 = \pi^2 r^2 (r^2 + h^2)$$

$$\Rightarrow \quad C^2 = \pi^2 r^4 + \pi^2 r^2 h^2$$

Let $\quad C^2 = Z$

Then, $\quad Z = \pi^2 r^4 + \pi^2 r^2 h^2 \quad \ldots(ii)$

$$\Rightarrow \quad Z = \pi^2 r^4 + \pi^2 r^2 \left(\frac{3V}{\pi r^2}\right)^2 \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad Z = \pi^2 r^4 + \pi^2 r^2 \times \frac{9V^2}{\pi^2 r^4} \quad (1\tfrac{1}{2})$$

$$\Rightarrow \quad Z = \pi^2 r^4 + \frac{9V^2}{r^2}$$

On differentiating both sides w.r.t. r, we get

$$\frac{dZ}{dr} = 4\pi^2 r^3 - \frac{18V^2}{r^3} \quad (1/2)$$

For maxima or minima, put $\frac{dZ}{dr} = 0$

$$\Rightarrow 4\pi^2 r^3 - \frac{18V^2}{r^3} = 0 \Rightarrow 4\pi^2 r^3 = \frac{18V^2}{r^3}$$

$$\Rightarrow \quad 4\pi^2 r^6 = 18\left(\frac{1}{3}\pi r^2 h\right)^2 \quad \left[\because V = \frac{1}{3}\pi r^2 h\right]$$

$$\Rightarrow \quad 4\pi^2 r^6 = 18 \times \frac{1}{9}\pi^2 r^4 h^2$$

$$\Rightarrow \quad 4\pi^2 r^6 = 2\pi^2 r^4 h^2$$

$$\Rightarrow \quad 2r^2 = h^2$$

$$\therefore \quad h = \sqrt{2} r$$

Hence, height $= \sqrt{2} \times$ (radius of base) **(1½)**

Also, $\quad \frac{d^2 Z}{dr^2} = \frac{d}{dr}\left(\frac{dZ}{dr}\right) = \frac{d}{dr}\left(4\pi^2 r^3 - \frac{18V^2}{r^3}\right)$

$$= 12\pi^2 r^2 + \frac{54V^2}{r^4}$$

$$\therefore \quad \frac{d^2 Z}{dr^2} = 12\pi^2 r^2 + \frac{54V^2}{r^4} > 0$$

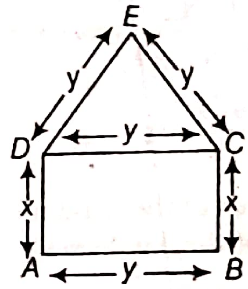
$\Rightarrow Z$ is minimum $\Rightarrow C$ is minimum.

Hence, curved surface area is least, when $h = \sqrt{2} r$.

Hence proved. (1½)

NOTE If C^2 is maximum/minimum, then C is also maximum/minimum.

30. Let $ABCD$ be the rectangle which is surmounted by an equilateral ΔEDC.



Now, given that

Perimeter of window = 12 m

$$\Rightarrow \quad 2x + 2y + y = 12$$

$$\therefore \quad x = 6 - \frac{3}{2}y \quad \ldots(i) \quad (1)$$

Let A denotes the combined area of the window.

Then, $\quad A$ = area of rectangle + area of equilateral triangle

$$\Rightarrow \quad A = xy + \frac{\sqrt{3}}{4}y^2$$

$$\Rightarrow \quad A = y\left(6 - \frac{3}{2}y\right) + \frac{\sqrt{3}}{4}y^2 \quad (1)$$

$$\left[\because x = 6 - \frac{3}{2}y \text{ from Eq. (i)}\right]$$

$$\Rightarrow \quad A = 6y - \frac{3}{2}y^2 + \frac{\sqrt{3}}{4}y^2$$

On differentiating both sides w.r.t. y, we get

$$\frac{dA}{dy} = 6 - 3y + \frac{\sqrt{3}}{2}y \quad (1)$$

For maxima or minima, put $\frac{dA}{dy} = 0$

$$\Rightarrow \quad 6 - 3y + \frac{\sqrt{3}}{2}y = 0$$

$$\Rightarrow \quad y\left(\frac{\sqrt{3}}{2} - 3\right) = -6$$

$$\Rightarrow \quad y = \frac{12}{6 - \sqrt{3}} \quad (1/2)$$

Now, $\quad \frac{d^2 A}{dy^2} = \frac{d}{dy}\left(\frac{dA}{dy}\right) = \frac{d}{dy}\left(6 - 3y + \frac{\sqrt{3}}{2}y\right)$

$$= -3 + \frac{\sqrt{3}}{2}$$

$$= \frac{-6 + \sqrt{3}}{2} < 0$$

$\therefore A$ is maximum. **(1)**

Now, on putting $y = \frac{12}{6-\sqrt{3}}$ in Eq. (1), we get

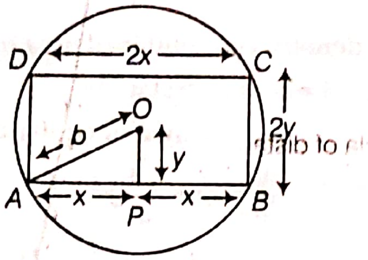
$$x = 6 - \frac{3}{2}\left(\frac{12}{6-\sqrt{3}}\right) \Rightarrow x = \frac{36 - 6\sqrt{3} - 18}{6-\sqrt{3}}$$

$$\therefore \quad x = \frac{18 - 6\sqrt{3}}{6-\sqrt{3}} \quad (1)$$

Hence the area of the window is largest when the dimensions of the window are

$$x = \frac{18-6\sqrt{3}}{6-\sqrt{3}} \text{ and } y = \frac{12}{6-\sqrt{3}} \quad (1/2)$$

31. Let $ABCD$ be the rectangle which is inscribed in a fixed circle whose centre is O and radius b. Let $AB = 2x$ and $BC = 2y$.



In right angled ΔOPA, by Pythagoras theorem, we have

$$AP^2 + OP^2 = OA^2$$

$$\Rightarrow x^2 + y^2 = b^2$$

$$\Rightarrow y^2 = b^2 - x^2$$

$$\Rightarrow y = \sqrt{b^2 - x^2} \quad \text{...(i) (1)}$$

Let A be the area of rectangle.

$$\therefore \quad A = (2x)(2y)$$

[$\because$ area of rectangle = length × breadth]

$$\Rightarrow A = 4xy$$

$$\Rightarrow A = 4x\sqrt{b^2 - x^2} \quad [\because y = \sqrt{b^2 - x^2}] \quad (1)$$

On differentiating both sides w.r.t. x, we get

$$\frac{dA}{dx} = 4x \cdot \frac{d}{dx}\sqrt{b^2 - x^2} + \sqrt{b^2 - x^2} \cdot \frac{d}{dx}(4x)$$

[by using product rule of derivative]

$$\Rightarrow \frac{dA}{dx} = 4x \cdot \frac{-2x}{2\sqrt{b^2 - x^2}} + \sqrt{b^2 - x^2} \cdot 4$$

$$= 4\left[\frac{b^2 - x^2 - x^2}{\sqrt{b^2 - x^2}}\right]$$

$$\Rightarrow \frac{dA}{dx} = 4\left(\frac{b^2 - 2x^2}{\sqrt{b^2 - x^2}}\right) \quad (1)$$

For maxima or minima, put $\frac{dA}{dx} = 0$

$$\therefore \quad 4\left(\frac{b^2 - 2x^2}{\sqrt{b^2 - x^2}}\right) = 0$$

$$\Rightarrow b^2 - 2x^2 = 0 \Rightarrow 2x^2 = b^2$$

$$\Rightarrow x = \frac{b}{\sqrt{2}} \quad (1/2)$$

[$\because x$ cannot be negative]

Also, $\frac{d^2A}{dx^2} = \frac{d}{dx}\left(\frac{dA}{dx}\right) = \frac{d}{dx}\left[\frac{4(b^2 - 2x^2)}{\sqrt{b^2 - x^2}}\right]$

$$\Rightarrow \frac{d^2A}{dx^2} = \frac{d}{dx}[4(b^2 - 2x^2)(b^2 - x^2)^{-1/2}]$$

$$\Rightarrow \frac{d^2A}{dx^2} = 4\Big[-4x(b^2 - x^2)^{-1/2} + (b^2 - 2x^2)\left(-\frac{1}{2}\right)(b^2 - x^2)^{-3/2}(-2x)\Big]$$

[by using product rule of derivative]

$$\Rightarrow \frac{d^2A}{dx^2} = 4\left[\frac{-4x}{\sqrt{b^2 - x^2}} + \frac{x(b^2 - 2x^2)}{(b^2 - x^2)^{3/2}}\right] \quad (1)$$

On putting $x = \frac{b}{\sqrt{2}}$, we get

$$\frac{d^2A}{dx^2} = 4\left[\frac{\frac{-4b}{\sqrt{2}}}{\sqrt{b^2 - \frac{b^2}{2}}} + \frac{\frac{b}{\sqrt{2}}\left(b^2 - 2\times\frac{b^2}{2}\right)}{\left(b^2 - \frac{b^2}{2}\right)^{3/2}}\right]$$

$$= 4\left[\frac{\frac{-4b}{\sqrt{2}}}{\sqrt{\frac{b^2}{2}}} + 0\right]$$

$$\Rightarrow \frac{d^2A}{dx^2} = -16 < 0$$

$\because \frac{d^2A}{dx^2} < 0$. So, A is maximum at $x = \frac{b}{\sqrt{2}}$. (1)

Now, on putting $x = \frac{b}{\sqrt{2}}$ in Eq. (i), we get

$$y = \sqrt{b^2 - \frac{b^2}{2}} = \sqrt{\frac{b^2}{2}} = \frac{b}{\sqrt{2}}$$

$$\therefore \quad x = y = \frac{b}{\sqrt{2}} \Rightarrow 2x = 2y = \sqrt{2}b$$

Hence, area of rectangle is maximum, when $2x = 2y$, i.e. when rectangle is a square. (1/2)

Hence proved.

32. Let x and y be the lengths of two sides of a rectangle. Again, let P denotes its perimeter and A be the area of rectangle.

Then, $P = 2(x + y)$ **(1)**

[$\because$ perimeter of rectangle $= 2(l + b)$]

$\Rightarrow \quad P = 2x + 2y$

$\Rightarrow \quad y = \dfrac{P - 2x}{2}$ ...(i) **(1)**

We know that, area of rectangle is given by

$A = xy$

$\Rightarrow \quad A = x\left(\dfrac{P - 2x}{2}\right)$ [by using Eq. (i)]

$\Rightarrow \quad A = \dfrac{Px - 2x^2}{2}$ **(1)**

On differentiating both sides w.r.t. x, we get

$$\frac{dA}{dx} = \frac{P - 4x}{2} \quad \textbf{(1)}$$

For maxima or minima, put $\dfrac{dA}{dx} = 0$

$\Rightarrow \quad \dfrac{P - 4x}{2} = 0 \Rightarrow P = 4x$

$\Rightarrow \quad 2x + 2y = 4x$ [$\because P = 2x + 2y$]

$\Rightarrow \quad x = y$

So, the rectangle is a square. **(1)**

Also, $\dfrac{d^2A}{dx^2} = \dfrac{d}{dx}\left(\dfrac{P - 4x}{2}\right)$

$= -\dfrac{4}{2} = -2 < 0$

$\Rightarrow A$ is maximum.

Hence, area is maximum, when rectangle is a square. **Hence proved.** **(1)**

33. Let x and y be the lengths of sides of a rectangle. Again, let A denotes its area and P be the perimeter.

Now, area of rectangle, $A = xy$

$\Rightarrow \quad y = \dfrac{A}{x}$ ...(i) **(1)**

And $\quad P = 2(x + y)$

[$\because$ perimeter of rectangle $= 2(l + b)$]

$\Rightarrow \quad P = 2\left(x + \dfrac{A}{x}\right)$

$\left[\because y = \dfrac{A}{x}, \text{ from Eq. (i)}\right]$ **(1)**

On differentiating both sides w.r.t. x, we get

$$\frac{dP}{dx} = 2\left(1 - \frac{A}{x^2}\right) \quad \textbf{(1)}$$

For maxima or minima, put $\dfrac{dP}{dx} = 0$.

$\Rightarrow \quad 2\left(1 - \dfrac{A}{x^2}\right) = 0 \Rightarrow 1 = \dfrac{A}{x^2}$ **(1)**

$\Rightarrow \quad A = x^2$

$\Rightarrow \quad xy = x^2$ [$\because A = xy$]

$\therefore \quad x = y$ **(1)**

Also, $\dfrac{d^2P}{dx^2} = \dfrac{d}{dx}\left[2\left(1 - \dfrac{A}{x^2}\right)\right] = 2\left(\dfrac{2A}{x^3}\right) = \dfrac{4A}{x^3} > 0$

Here, x and A being the side and area of rectangle can never be negative. So, P is minimum.

Hence, perimeter of rectangle is minimum, when rectangle is a square. **(1)**

Hence proved.

34. Do same as Q. No. 12.

35.

> First, consider any point on the curve, use the formula of distance between two points. Then, square both sides and eliminate one variable with the help of given equation. Further, apply concept of maxima and minima to find the required point.

The given equation of curve is $y^2 = 2x$ and the given point is $Q(1, 4)$.

Let $P(x, y)$ be any point on the curve. **(1)**

Now, distance between points P and Q is given by

$$PQ = \sqrt{(1 - x)^2 + (4 - y)^2}$$

$\left[\text{using distance formula, } d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\right]$

$\Rightarrow \quad PQ = \sqrt{1 + x^2 - 2x + 16 + y^2 - 8y}$

$= \sqrt{x^2 + y^2 - 2x - 8y + 17}$

On squaring both sides, we get

$PQ^2 = x^2 + y^2 - 2x - 8y + 17$

$\Rightarrow \quad PQ^2 = \left(\dfrac{y^2}{2}\right)^2 + y^2 - 2\left(\dfrac{y^2}{2}\right) - 8y + 17$

$\left[\text{given, } y^2 = 2x \Rightarrow x = \dfrac{y^2}{2}\right]$

$\therefore \quad PQ^2 = \dfrac{y^4}{4} + y^2 - y^2 - 8y + 17$

$\Rightarrow \quad PQ^2 = \dfrac{y^4}{4} - 8y + 17$ **(1)**

Let $PQ^2 = Z$

Then, $Z = \frac{y^4}{4} - 8y + 17$

On differentiating both sides w.r.t. y, we get

$$\frac{dZ}{dy} = \frac{4y^3}{4} - 8 = y^3 - 8 \quad (1)$$

For maxima or minima, put $\frac{dZ}{dy} = 0$

$\Rightarrow \quad y^3 - 8 = 0 \Rightarrow y^3 = 8$

$\Rightarrow \quad y = 2$ (1)

Also, $\frac{d^2Z}{dy^2} = \frac{d}{dy}(y^3 - 8) = 3y^2$

On putting $y = 2$, we get

$$\left(\frac{d^2Z}{dy^2}\right)_{y=2} = 3(2)^2 = 12 > 0$$

$\because \quad \frac{d^2Z}{dy^2} > 0$

$\therefore Z$ is minimum and therefore PQ is also minimum as $Z = PQ^2$. (1)

On putting $y = 2$ in the given equation, i.e. $y^2 = 2x$, we get

$(2)^2 = 2x$

$\Rightarrow \quad 4 = 2x$

$\Rightarrow \quad x = 2$

Hence, the point which is at a minimum distance from point $(1, 4)$ is $P(2, 2)$. (1)

36. First, find length of circular part (or its circumference) and calculate the length of square part (or its perimeter). Add these two terms and equate it to 28 m and apply second derivative test to check minimum.

Let x m be the side of the square and r be the radius of circular part. Then,

Length of square part = perimeter of square
= 4 × Side = $4x$

and length of circular part
= circumference of circle = $2\pi r$

Given, length of wire = 28

$\Rightarrow \quad 4x + 2\pi r = 28$

$\Rightarrow \quad 2x + \pi r = 14$

$\therefore \quad x = \frac{14 - \pi r}{2}$ ...(i) (1)

Let A denotes the combined area of circle and square.

Then, $A = \pi r^2 + x^2$

$\Rightarrow \quad A = \pi r^2 + \left(\frac{14 - \pi r}{2}\right)^2$

$\left[\because x = \frac{14 - \pi r}{2}, \text{ from Eq. (i)}\right]$

On differentiating both sides w.r.t. r, we get

$$\frac{dA}{dr} = 2\pi r + 2\left(\frac{14 - \pi r}{2}\right)\left(-\frac{\pi}{2}\right)$$

$$= 2\pi r - \left(\frac{14\pi - \pi^2 r}{2}\right) \quad (1)$$

For maxima or minima, put $\frac{dA}{dr} = 0$

$\Rightarrow \quad 2\pi r - \left(\frac{14\pi - \pi^2 r}{2}\right) = 0$

$\Rightarrow \quad 2\pi r = \frac{14\pi - \pi^2 r}{2}$

$\Rightarrow \quad r = \frac{14}{\pi + 4}$ (1)

Also, $\frac{d^2A}{dr^2} = \frac{d}{dr}\left(\frac{dA}{dr}\right)$

$$= \frac{d}{dr}\left[2\pi r - \left(\frac{14\pi - \pi^2 r}{2}\right)\right]$$

$\Rightarrow \quad \frac{d^2A}{dr^2} = 2\pi + \frac{\pi^2}{2} > 0$

Thus, $\frac{d^2A}{dr^2} > 0 \Rightarrow A$ is minimum. (1)

Now, on putting $r = \frac{14}{\pi + 4}$ in Eq. (i), we get

$$x = \frac{14 - \pi\left(\frac{14}{\pi + 4}\right)}{2}$$

$$= \frac{14\pi + 56 - 14\pi}{2(\pi + 4)} = \frac{28}{\pi + 4}$$

$\therefore \quad x = \frac{28}{\pi + 4}$ and $r = \frac{14}{\pi + 4}$ (1)

Now, length of circular part

$$= 2\pi r = 2\pi \times \frac{14}{\pi + 4} = \frac{28\pi}{\pi + 4}$$

and length of square part $= 4x = 4 \times \frac{28}{\pi + 4} = \frac{112}{\pi + 4}$

which are the required length of two pieces. (1)

37. Do same as Q. No. 6.

38. Given equation of ellipse is

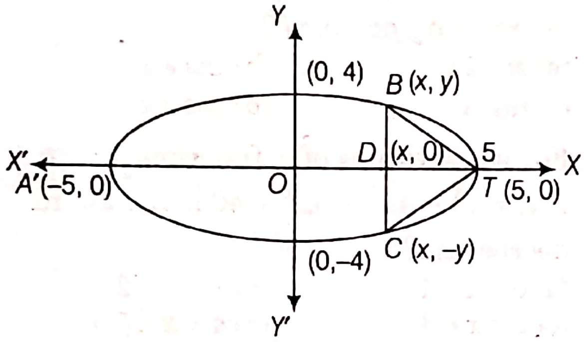
$$\frac{x^2}{25}+\frac{y^2}{16}=1.$$

Here, $a=5, b=4$

$\therefore \quad a>b$

So, major axis is along X-axis.

Let ΔBTC be the isosceles triangle which is inscribed in the ellipse and $OD = x$, $BC = 2y$ and $TD = 5-x$.



Let A denotes the area of triangle. Then, we have

$$A = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times BC \times TD$$

$$\Rightarrow \quad A = \frac{1}{2} \cdot 2y\,(5-x) \Rightarrow A = y\,(5-x) \quad \textbf{(1)}$$

On squaring both sides, we get

$$A^2 = y^2\,(5-x)^2 \quad \ldots(\text{i})$$

Now, $\dfrac{x^2}{25}+\dfrac{y^2}{16}=1$

$\Rightarrow \quad \dfrac{y^2}{16} = 1 - \dfrac{x^2}{25}$

$\Rightarrow \quad y^2 = \dfrac{16}{25}(25-x^2)$

On putting value of y^2 in Eq. (i), we get

$$A^2 = \frac{16}{25}(25-x^2)(5-x)^2$$

Let $\quad A^2 = Z$

Then, $\quad Z = \dfrac{16}{25}(25-x^2)(5-x)^2 \quad$ **(1)**

On differentiating both sides w.r.t. x, we get

$$\frac{dZ}{dx} = \frac{16}{25}[(25-x^2)\,2(5-x)(-1) + (5-x)^2(-2x)]$$

[by using product rule of derivative]

$$= \frac{16}{25}(-2)(5-x)^2(2x+5)$$

$$= \frac{-32}{25}(5-x)^2(2x+5) \quad \textbf{(1)}$$

For maxima or minima, put $\dfrac{dZ}{dx} = 0$

$\Rightarrow -\dfrac{32}{25}(5-x)^2(2x+5) = 0 \Rightarrow x = 5 \text{ or } -\dfrac{5}{2}$

Now, when $x = 5$, then

$$Z = \frac{16}{25}(25-25)(5-5)^2 = 0$$

which is not possible.

So, $x = 5$ is rejected.

$\therefore \quad x = -\dfrac{5}{2} \quad$ **(1)**

Now, $\dfrac{d^2Z}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dZ}{dx}\right) = \dfrac{d}{dx}\left[-\dfrac{32}{25}(5-x)^2(2x+5)\right]$

$$= -\frac{32}{25}[(5-x)^2 \cdot 2 - (2x+5)\,2(5-x)]$$

$$= -\frac{64}{25}(5-x)(-3x) = \frac{192x}{25}(5-x)$$

$\therefore$ At $x = \dfrac{-5}{2}$, $\left(\dfrac{d^2Z}{dx^2}\right)_{x=-\frac{5}{2}} < 0$

$\Rightarrow Z$ is maximum. **(1)**

$\therefore$ Area A is maximum, when $x = -\dfrac{5}{2}$ and $y = 12$

Clearly,

$$Z = A^2 = \frac{16}{25}\left(25 - \frac{25}{4}\right)\left[5+\frac{5}{2}\right]^2$$

$$= \frac{16}{25} \times \frac{75}{4} \times \frac{225}{4} = 3 \times 225$$

$\therefore$ The maximum area, $A = \sqrt{3 \times 225}$

$= 15\sqrt{3}$ sq units **(1)**

NOTE If A^2 is maximum/minimum, then A is also maximum/minimum.

39. Let V be the volume, S be the total surface area of a right circular cylinder which is open at the top. Again, let r be the radius of base and h be the height.

Now, $S = 2\pi rh + \pi r^2$ [$\because$ cylinder is open at top]

$\Rightarrow \quad h = \dfrac{S - \pi r^2}{2\pi r} \quad \ldots$(i) **(1)**

Also, volume of cylinder is given by

$$V = \pi r^2 h$$

$\Rightarrow \quad V = \pi r^2\left(\dfrac{S-\pi r^2}{2\pi r}\right) \quad$ [using Eq. (i)]

$\Rightarrow \quad V = \dfrac{rS - \pi r^3}{2} \quad$ **(1)**

On differentiating both sides w.r.t. r, we get

$$\frac{dV}{dr} = \frac{S - 3\pi r^2}{2}$$ (1)

For maxima or minima, put $\frac{dV}{dx} = 0$

$\therefore \quad \frac{S - 3\pi r^2}{2} = 0 \Rightarrow S = 3\pi r^2$

$\Rightarrow \quad 2\pi rh + \pi r^2 = 3\pi r^2$ [from Eq. (i)]

$\Rightarrow \quad h = r$ (1)

$\therefore$ Height of cylinder = Radius of the base

Also, $\frac{d^2V}{dr^2} = \frac{d}{dr}\left(\frac{dV}{dr}\right) = \frac{d}{dr}\left(\frac{S - 3\pi r^2}{2}\right) = -\frac{6\pi r}{2}$

$= -3\pi r < 0$, as $r > 0$ (1)

Thus, $\frac{d^2V}{dr^2} < 0 \Rightarrow V$ is maximum.

Hence, volume of cylinder is maximum, when its height is equal to radius of the base. (1)

Hence proved.

Objective Questions

(For Complete Chapter)

1 Mark Questions

1. A sphere increases its volume at the rate of π cm^3/ s. The rate at which its surface area increases, when the radius is 1 cm is

(a) 2π sq cm/s (b) π sq cm/s

(c) $\frac{3\pi}{2}$ sq cm/s (d) $\frac{\pi}{2}$ sq cm/s

2. If gas is being pumped into a spherical balloon at the rate of 30 ft^3/min. Then, the rate at which the radius increases, when it reaches the value 15 ft is

(a) $\frac{1}{15\pi}$ ft/min (b) $\frac{1}{30\pi}$ ft/min

(c) $\frac{1}{20}$ ft/min (d) $\frac{1}{25}$ ft/min

3. The radius of a cylinder is increasing at the rate of 3 m/s and its altitude is decreasing at the rate of 4 m/s. The rate of change of volume, when radius is 4 m and altitude is 6 m, is

(a) 80 π cu m/s (b) 144 π cu m/s

(c) 80 cu m/s (d) 64 cu m/s

4. The length of the longest interval, in which $f(x) = 3 \sin x - 4 \sin^3 x$ is increasing, is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{2}$ (d) π

5. Which of the following function is decreasing on $(0, \pi/2)$?

(a) $\sin 2x$ (b) $\cos 3x$

(c) $\tan x$ (d) $\cos 2x$

6. For what values of x, function $f(x) = x^4 - 4x^3 + 4x^2 + 40$ is monotonic decreasing?

(a) $0 < x < 1$ (b) $1 < x < 2$

(c) $2 < x < 3$ (d) $4 < x < 5$

7. If $y = 2x^3 - 2x^2 + 3x - 5$, then for $x = 2$ and $\Delta x = 0.1$, value of Δy is

(a) 2.002 (b) 1.9

(c) 0 (d) 0.9

8. If the error committed in measuring the radius of the circle is 0.05%, then the corresponding error in calculating the area is

(a) 0.05% (b) 0.0025%

(c) 0.25% (d) 0.1%

9. The slope of the normal to the curve $y = x^2 - \frac{1}{x^2}$ at $(-1, 0)$ is

(a) $\frac{1}{4}$ (b) $-\frac{1}{4}$

(c) 4 (d) -4

10. The point of the parabola $y^2 = 64x$ which is nearest to the line $4x + 3y + 35 = 0$ has coordinates

(a) $(9, -24)$ (b) $(1, 81)$

(c) $(4, -16)$ (d) $-(-9, -24)$

11. The minimum radius vector of the curve $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 1$ is of length

(a) $a - b$ (b) $a + b$

(c) $2a + b$ (d) None of these

12. The condition that $f(x) = ax^3 + bx^2 + cx + d$ has no extreme value is

(a) $b^2 > 3ac$ (b) $b^2 = 4ac$
(c) $b^2 = 3ac$ (d) $b^2 < 3ac$

13. The maximum value of xe^{-x} is

(a) e (b) $1/e$
(c) $-e$ (d) $-1/e$

14. The least value of the function $f(x) = ax + b/x, a > 0,$ $b > 0, x > 0$ is

(a) $\sqrt{ab}$ (b) $2\sqrt{\frac{a}{b}}$ (c) $2\sqrt{\frac{b}{a}}$ (d) $2\sqrt{ab}$

Solutions

1. (a) Let volume of sphere, $V = \frac{4}{3}\pi r^3$

$$\Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\Rightarrow \pi = 4\pi r^2 \frac{dr}{dt} \qquad \left[\because \frac{dV}{dt} = \pi\right]$$

$$\Rightarrow \frac{dr}{dt} = \frac{1}{4r^2} \qquad \ldots(i)$$

Now, $\frac{dS}{dt} = \frac{d}{dt}(4\pi r^2) = 4\pi\left(2r\frac{dr}{dt}\right)$

$$\therefore \left(\frac{dS}{dt}\right)_{r=1} = 4\pi\left(2\cdot1\cdot\frac{1}{4}\right) = 2\pi\,\text{cm}^2/\text{s}$$

[from Eq. (i)]

2. (b) Let $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2\frac{dr}{at}$

$$\therefore \frac{dr}{dt} = \frac{30}{4\times\pi\times15\times15} = \frac{1}{30\pi}\text{ ft/min}$$

$$\left[\because \frac{dV}{dt} = 30, r = 15\right]$$

3. (a) Let h and r be the height and radius of cylinder.

Given that, $\frac{dr}{dt} = 3\,\text{m/s}$, $\frac{dh}{dt} = -4\,\text{m/s}$

Let volume of cylinder, $V = \pi r^2 h$

$$\Rightarrow \frac{dV}{dt} = \pi\left[r^2\frac{dh}{dt} + h\cdot2r\frac{dr}{dt}\right]$$

At $r = 4\,\text{m}$ and $h = 6\,\text{m}$,

$$\frac{dV}{dt} = \pi[-64 + 144] = 80\pi\text{ cu m/s}$$

4. (a) Let $f(x) = 3\sin x - 4\sin^3 x = \sin 3x$

Since, $\sin x$ is increasing in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$$\therefore -\frac{\pi}{2} \le 3x \le \frac{\pi}{2} \Rightarrow -\frac{\pi}{6} \le x \le \frac{\pi}{6}$$

Thus, the length of interval $= \left[\frac{\pi}{6} - \left(-\frac{\pi}{6}\right)\right] = \frac{\pi}{3}$

5. (d) $f(x) = \cos 2x$, $f'(x) = -2\sin 2x < 0$ in $\left(0, \frac{\pi}{2}\right)$

So, $\cos 2x$ is decreasing in $\left(0, \frac{\pi}{2}\right)$

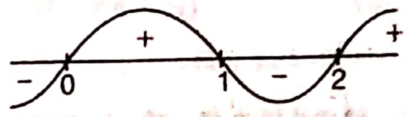
6. (b) $\because f(x) = x^4 - 4x^3 + 4x^2 + 40$

$$\Rightarrow f'(x) = 4x^3 - 12x^2 + 8x$$

For monotonic decreasing, $f'(x) < 0$

$$\Rightarrow x(4x^2 - 12x + 8) < 0$$

$$\Rightarrow x(x^2 - 3x + 2) < 0 \Rightarrow x(x-1)(x-2) < 0$$



$$\therefore x \in (-\infty, 0) \cup (1, 2)$$

7. (b) We have, $y = 2x^3 - 2x^2 + 3x - 5$

Now, $\frac{dy}{dx} = 6x^2 - 4x + 3$

$$\therefore \left(\frac{dy}{dx}\right)_{x=2} = 24 - 8 + 3 = 19$$

$$\because \Delta y = \left(\frac{dy}{dx}\right)_{x=2} \Delta x$$

$$\Rightarrow \Delta y = 19 \times (0.1) = 1.9$$

8. (d) We know that, area of circle, $A = \pi r^2$

Taking log on both sides, we get

$$\log A = \log\pi + 2\log r$$

$$\therefore \frac{\Delta A}{A}\times100 = 2\times\frac{\Delta r}{r}\times100 = 2\times0.05 = 0.1\%$$

9. (a) Given curve is $y = x^2 - 1/x^2$.

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 2x + \frac{2}{x^3}$$

At point $(-1, 0)$, $\frac{dy}{dx} = 2(-1) + \frac{2}{(-1)^3} = -4$

$\therefore$ Slope of normal to the curve $= -\frac{1}{dy/dx}$

$$= \frac{1}{-4} = \frac{1}{4}$$

10. (*a*) Given, equation of parabola is $y^2 = 64x$ ...(i)

Since, the point at which the tangent to the curve is parallel to the line is the nearest point on the curve.

On differentiating both sides of Eq. (i), we get

$$2y\frac{dy}{dx} = 64 \Rightarrow \frac{dy}{dx} = \frac{32}{y} \Rightarrow \frac{32}{y} = -\frac{4}{3}$$

$\therefore$ $y = -24$ and $x = 9$

11. (*b*) Given, curve is $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 1.$

Let radius vector be *r*.

$\therefore$ $r^2 = x^2 + y^2$

$\Rightarrow$ $r^2 = \frac{a^2y^2}{y^2 - b^2} + y^2$ $\left[\because \frac{a^2}{x^2} + \frac{b^2}{y^2} = 1\right]$

For minimum value of *r*,

$$\frac{d(r^2)}{dy} = 0 \Rightarrow \frac{-2y\,b^2a^2}{(y^2 - b^2)^2} + 2y = 0$$

$\Rightarrow$ $y^2 = b(a + b)$

$\therefore$ $x^2 = a(a + b) \Rightarrow r^2 = (a + b)^2$

$\Rightarrow$ $r = a + b$

12. (*d*) Given, curve is $f(x) = ax^3 + bx^2 + cx + d$

On differentiating w.r.t. *x*, we get

$$f'(x) = 3ax^2 + 2bx + c$$

For extremum, $f'(x) = 0 \Rightarrow 3ax^2 + 2bx + c = 0$

Since, it has no extremum value.

$\therefore$ $b^2 - 4ac < 0$

$\Rightarrow$ $(2b)^2 - 4 \times 3a \times c < 0$

$\Rightarrow$ $4b^2 - 12ac < 0$

$\Rightarrow$ $b^2 - 3ac < 0$

$\Rightarrow$ $b^2 < 3ac$

13. (*b*) Let $f(x) = xe^{-x}$

$\therefore$ $f'(x) = -xe^{-x} + e^{-x} = e^{-x}(1 - x)$

For maxima or minima, put $f'(x) = 0$

$\Rightarrow$ $e^{-x}(1 - x) = 0 \Rightarrow x = 1$ $[\because e^{-x} > 0, \forall x \in R]$

Further, $f''(x) < 0$ at $x = 1$

Therefore, $f(x)$ attains its maxima at $x = 1$ and the maximum value is $\frac{1}{e}$.

14. (*d*) Given, $f(x) = ax + \frac{b}{x}$, $a, b, x > 0$

$\Rightarrow$ $f'(x) = a - \frac{b}{x^2}$

For maxima or minima, put $f'(x) = 0$

$\Rightarrow$ $x^2 = \frac{b}{a}$, $x = \pm\sqrt{\frac{b}{a}}$

At $x = \sqrt{\frac{b}{a}}$, $\frac{d^2y}{dx^2} > 0$

So, $f(x)$ attains its minima at $x = \sqrt{b/a}$ and the minimum value is $2\sqrt{ab}$.