Solutions

1. $\int_2^3 3^x\,dx = \left(\frac{3^x}{\log 3}\right)_2^3 = \frac{1}{\log 3}[3^x]_2^3 = \frac{1}{\log 3}[3^3 - 3^2]$

$= \frac{1}{\log 3}(27-9) = \frac{18}{\log 3}$ **[1]**

2. $\int_0^{\pi/4} \tan x\,dx = [\log|\sec x|]_0^{\pi/4}$

$= \log\left|\sec\frac{\pi}{4}\right| - \log|\sec 0|$

$= \log|\sqrt{2}| - \log|1|$

$= \frac{1}{2}\log 2$ **(1)**

3. Let $I = \int_0^1 x e^{x^2}\,dx$

Put $x^2 = t \Rightarrow 2x dx = dt \Rightarrow dx = \frac{dt}{2x}$

Lower limit when $x = 0$, then $t = 0$

Upper limit when $x = 1$, then $t = 1$.

$\therefore \quad I = \int_0^1 x e^t \frac{dt}{2x} = \frac{1}{2}\int_0^1 e^t dt$

$= \frac{1}{2}[e^t]_0^1 = \frac{1}{2}[e^1 - e^0] = \frac{1}{2}[e-1]$ **(1)**

4. Let $I = \int_0^{\pi/4} \sin 2x\,dx$

$= \left[\frac{-\cos 2x}{2}\right]_0^{\pi/4} = -\frac{1}{2}[\cos 2x]_0^{\pi/4}$

$= -\frac{1}{2}\left[\cos 2\frac{\pi}{4} - \cos 0\right]$

$= -\frac{1}{2}\left[\cos\frac{\pi}{2} - 1\right] = -\frac{1}{2}[0-1] = \frac{1}{2}$ **(1)**

5. Let $I = \int_0^1 \frac{1}{\sqrt{1-x^2}}dx$

$= [\sin^{-1} x]_0^1 \qquad \left[\because \int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + C\right]$

$= \sin^{-1} 1 - \sin^{-1} 0$

$= \sin^{-1}\left(\sin\frac{\pi}{2}\right) - \sin^{-1}(\sin 0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$ **(1)**

6. Given, $\int_0^a \frac{1}{4+x^2}dx = \frac{\pi}{8}$ ...(i)

Now, consider $I = \int_0^a \frac{1}{x^2 + (2)^2}dx$

$\Rightarrow \quad I = \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^a$

$\left[\because \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C\right]$

$\Rightarrow \quad I = \frac{1}{2}\tan^{-1}\frac{a}{2} - \frac{1}{2}\tan^{-1}(0)$

$\Rightarrow \quad I = \frac{1}{2}\tan^{-1}\frac{a}{2}$ ...(ii) **(1/2)**

From Eqs. (i) and (ii), we get

$\frac{1}{2}\tan^{-1}\frac{a}{2} = \frac{\pi}{8} \Rightarrow \tan^{-1}\frac{a}{2} = \frac{\pi}{4}$

$\Rightarrow \quad \frac{a}{2} = \tan\frac{\pi}{4}$

$\Rightarrow \quad \frac{a}{2} = 1 \Rightarrow a = 2 \qquad \left[\because \tan\frac{\pi}{4} = 1\right]$ **(1/2)**

7. Given, $f(x) = \int_0^x \underset{\text{I}}{t} \underset{\text{II}}{\sin t}\, dt$

$$= \left[t \int \sin t\, dt - \int \left[\frac{d}{dt}(t) \int \sin t\, dt \right] dt \right]_0^x$$

[using integration by parts]

$$= [t(-\cos t)]_0^x - \int_0^x (-\cos t)\, dt$$
$$= [-t \cos t]_0^x + [\sin t]_0^x$$
$$= -x\cos x + 0 + \sin x - 0$$
$$= \sin x - x\cos x$$

Thus, $f(x) = \sin x - x\cos x$ **(1/2)**

On differentiating both sides w.r.t. x, we get

$$f'(x) = \cos x - \left[x\frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(x) \right]$$

[by product rule of derivative]

$$= \cos x - [x(-\sin x) + \cos x]$$
$$= \cos x + x\sin x - \cos x = x\sin x$$ **(1/2)**

8. Let $I = \int_2^4 \frac{x}{x^2+1}\, dx$

Put $x^2 + 1 = t \Rightarrow 2x\, dx = dt \Rightarrow x\, dx = \frac{dt}{2}$

Lower limit when $x = 2$, then $t = 2^2 + 1 = 5$

Upper limit when $x = 4$, then $t = 4^2 + 1 = 17$. **(1/2)**

$$\therefore \quad I = \int_5^{17} \frac{1}{t}\frac{dt}{2} = \frac{1}{2}\int_5^{17} \frac{1}{t}\, dt = \frac{1}{2}[\log|t|]_5^{17}$$

$$= \frac{1}{2}[\log 17 - \log 5] = \frac{1}{2}\log\left(\frac{17}{5}\right)$$

$\left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$ **(1/2)**

9. Let $I = \int_0^3 \frac{dx}{9 + x^2} \Rightarrow I = \int_0^3 \frac{dx}{x^2 + (3)^2}$

$$\Rightarrow I = \left[\frac{1}{3}\tan^{-1}\frac{x}{3}\right]_0^3 \left[\because \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C\right]$$

$$\Rightarrow I = \frac{1}{3}\left[\tan^{-1}\left(\frac{3}{3}\right) - \tan^{-1}(0)\right]$$

$$= \frac{1}{3}[\tan^{-1}(1) - 0] = \frac{1}{3}\left(\frac{\pi}{4}\right) = \frac{\pi}{12}$$ **(1)**

10. Let $I = \int_0^{\pi/2} e^x (\sin x - \cos x)\, dx$

$$\Rightarrow \quad I = -\int_0^{\pi/2} e^x (\cos x - \sin x)\, dx$$

Now, consider, $f(x) = \cos x$

then $f'(x) = -\sin x$

Now, by using $\int e^x [f(x) + f'(x)]\, dx = e^x f(x) + C$,

we get $I = -[e^x \cos x]_0^{\pi/2}$

$$= -e^{\pi/2}\cos\frac{\pi}{2} + e^0 \cos(0)$$
$$= 0 + 1(1) = 1$$ **(1)**

11. Let $I = \int_e^{e^2} \frac{dx}{x \log x}$

Put $\log x = t \Rightarrow \frac{1}{x}dx = dt$

Lower limit when $x = e$, then $t = \log e = 1$

Upper limit when $x = e^2$, then $t = \log e^2 = 2$

$\therefore \quad I = \int_1^2 \frac{dt}{t} = [\log|t|]_1^2 = \log 2 - \log 1 = \log 2$ **(1)**

12. Let $I = \int_0^1 \frac{\tan^{-1} x}{1+x^2}dx$

Put $\tan^{-1} x = t \Rightarrow \frac{1}{1+x^2}dx = dt$

Lower limit when $x = 0$, then $t = 0$

Upper limit when $x = 1$, then $t = \pi/4$.

$$\therefore \quad I = \int_0^{\pi/4} t\, dt = \left[\frac{t^2}{2}\right]_0^{\pi/4} = \frac{1}{2}\left[\left(\frac{\pi}{4}\right)^2 - (0)^2\right] = \frac{\pi^2}{32}$$ **(1)**

13. Let $I = \int_1^2 \frac{x^3 - 1}{x^2}dx = \int_1^2 \left(x - \frac{1}{x^2}\right)dx$

$$= \left[\frac{x^2}{2} + \frac{1}{x}\right]_1^2 = \left(\frac{(2)^2}{2} + \frac{1}{2}\right) - \left(\frac{(1)^2}{2} + \frac{1}{1}\right)$$

$$= \left(2 + \frac{1}{2}\right) - \left(\frac{1}{2} + 1\right) = 1$$ **(1)**

14. $\int_2^3 \frac{1}{x}dx = [\log|x|]_2^3 = \log 3 - \log 2 = \log\frac{3}{2}$

$\left[\because \log m - \log n = \log\frac{m}{n}\right]$ **(1)**

15. $\int_0^2 \sqrt{4 - x^2}\, dx = \left[\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$

$$\left[\because \int \sqrt{a^2 - x^2}dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C\right]$$

$$= 0 + 2\sin^{-1} 1 = 2\sin^{-1}\left(\sin\frac{\pi}{2}\right) = 2 \times \frac{\pi}{2} = \pi$$ **(1)**

16. Let $I = \int_0^1 \frac{e^x}{1 + e^{2x}}dx = \int_0^1 \frac{e^x}{1 + (e^x)^2}dx$

Put $e^x = t \Rightarrow e^x\, dx = dt$

Also, when $x = 0$, then $t = 1$ and when $x = 1$, then $t = e$ **(1/2)**

Now, $I = \int_1^e \frac{dt}{1+t^2} = (\tan^{-1} t)_1^e$

$= \tan^{-1} e - \tan^{-1} 1 = \tan^{-1}\left(\frac{e-1}{1+e}\right)$ **(1/2)**

17. $\int_1^{\sqrt{3}} \frac{dx}{1+x^2} = [\tan^{-1} x]_1^{\sqrt{3}}$ $\left[\because \int \frac{dx}{1+x^2} = \tan^{-1} x\right]$

$= \tan^{-1}\sqrt{3} - \tan^{-1} 1$

$= \tan^{-1}\left(\tan\frac{\pi}{3}\right) - \tan^{-1}\left(\tan\frac{\pi}{4}\right)$

$\left[\because \sqrt{3} = \tan\frac{\pi}{3} \text{ and } 1 = \tan\frac{\pi}{4}\right]$

$= \frac{\pi}{3} - \frac{\pi}{4} = \frac{4\pi - 3\pi}{12} = \frac{\pi}{12}$ **(1)**

18. Do same as Q. No. 8. **[Ans. log 2]**

19. Do same as Q. No. 17. $\left[\textbf{Ans. } \frac{\pi}{4}\right]$

20. Use the property $\int_{-a}^{a} f(x)dx = 0$, if $f(x)$ is an odd function.

Let $I = \int_{-\pi/4}^{\pi/4} \sin^3 x \, dx$

Consider, $f(x) = \sin^3 x$. Then, $f(-x) = \sin^3(-x)$

$= (-\sin x)^3 = -\sin^3 x = -f(x)$

$\Rightarrow f(x)$ is an odd function. **(1/2)**

Thus, the given integrand is an odd function.

$\therefore \quad I = 0$

$\left[\because \int_{-a}^{a} f(x)\,dx = 0, \text{ if } f(x) \text{ is an odd function}\right]$ **(1/2)**

21. Do same as Q. No. 20. **[Ans. 0]**

22. Let $I = \int_{-\pi}^{\pi} (1 - x^2) \sin x \cos^2 x \, dx$

Again, let $f(x) = (1 - x^2) \sin x \cos^2 x$

$\therefore f(-x) = [1 - (-x)^2] \sin(-x) \cos^2(-x)$

$= (1 - x^2)(-\sin x) \cos^2 x$

$= -(1 - x^2) \sin x \cos^2 x$

$= -f(x)$ **(1)**

$\therefore f(x)$ is odd function

$\therefore \quad I = 0$

$[\because \int_{-a}^{a} f(x)\,dx = 0, \text{ if } f(x) \text{ is odd function}]$ **(1)**

23. Let, $I = \int_{-1}^{2} \frac{|x|}{x} dx = \int_{-1}^{0} \frac{|x|}{x} dx + \int_0^2 \frac{|x|}{x} dx$

$= \int_{-1}^{0} \frac{-x}{x} dx + \int_0^2 \frac{x}{x} dx$ $\left[\because |x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}\right]$

$= \int_{-1}^{0} -1 \, dx + \int_0^2 1 \, dx = [-x]_{-1}^{0} + [x]_0^2$

$= [0 - (1)] + [2 - 0] = -1 + 2 = 1$ **(2)**

24. To prove $\int_0^a f(x)\,dx = \int_0^a f(a - x)\,dx$

Consider RHS $= \int_0^a f(a - x)\,dx$

Putting $t = a - x$, then $dt = -dx$

Also, when $x = 0$, then $t = a$

and when $x = a$, then $t = 0$

Now, RHS $= -\int_a^0 f(t)dt = \int_0^a f(t)dt$

$= \int_0^a f(x)dx = \text{LHS}$ **Hence proved. (1)**

Now, let $I = \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$ ...(i)

$= \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)\,dx}{1 + \cos^2(\pi - x)}$

$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a - x)\,dx\right]$

$= \int_0^{\pi} \frac{(\pi - x)\sin x\,dx}{1 + \cos^2 x}$

$= \pi \int_0^{\pi} \frac{\sin x\,dx}{1 + \cos^2 x} - \int_0^{\pi} \frac{x \sin x\,dx}{1 + \cos^2 x}$ **(1)**

$I = \pi \int_0^{\pi} \frac{\sin x\,dx}{1 + \cos^2 x} - I$ [from Eq. (i)]

$\Rightarrow \quad 2I = \pi \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx$

$\Rightarrow \quad I = \frac{\pi}{2} \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx$ **(1)**

Putting $\cos x = t \Rightarrow -\sin x\,dx = dt$

Also, when $x = 0$, then $t = 1$ and when $x = \pi$, then $t = -1$

Now, $I = -\frac{\pi}{2} \int_1^{-1} \frac{dt}{1 + t^2} = \frac{\pi}{2} \int_{-1}^{1} \frac{dt}{1 + t^2}$

$= \frac{\pi}{2} [\tan^{-1} t]_{-1}^{1} = \frac{\pi}{2} [\tan^{-1}(1) - \tan^{-1}(-1)]$

$= \frac{\pi}{2}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{\pi}{2}\left[\frac{\pi}{2}\right] = \frac{\pi^2}{4}$ **(1)**

25. RHS $= \int_0^a f(a-x)\,dx$

Put $(a-x) = t \Rightarrow dx = -dt$

Now, when $x = 0$, we have $t = a$

and when $x = a$, we have $t = 0$

$$\therefore \int_0^a f(a-x)\,dx = -\int_a^0 f(t)\,dt$$

$$= \int_0^a f(t)\,dt = \int_0^a f(x)\,dx = \text{LHS}$$

Hence, $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Now, let $I = \int_0^{\pi/2} \dfrac{x}{(\sin x + \cos x)}\,dx$ ...(i) **(1)**

$$\text{Then, } I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right)}{\sin\left\{\left(\frac{\pi}{2}\right) - x\right\} + \cos\left\{\left(\frac{\pi}{2}\right) - x\right\}}\,dx$$

$$I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right)}{(\cos x + \sin x)}\,dx$$

$$= \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right)}{(\sin x + \cos x)}\,dx \quad \text{...(ii) (1)}$$

On adding Eqs. (i) and (ii), we get

$$2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{dx}{\sin x + \cos x}$$

$$\Rightarrow \quad I = \frac{\pi}{4}\int_0^{\pi/2} \frac{dx}{\sin x + \cos x}$$

$$= \frac{\pi}{4}\int_0^{\pi/2} \frac{dx}{\left[\dfrac{2\tan\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)} + \dfrac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}\right]}$$

$$= \frac{\pi}{4}\int_0^{\pi/2} \frac{\sec^2\left(\frac{x}{2}\right)}{1-\tan^2\left(\frac{x}{2}\right) + 2\tan\left(\frac{x}{2}\right)}\,dx \quad \text{(1)}$$

$$= \frac{\pi}{4}\int_0^{1} \frac{2\,dt}{(1-t^2+2t)}, \text{ where } t = \tan\frac{x}{2}$$

$$\left[x = 0 \Rightarrow t = 0 \text{ and } x = \frac{\pi}{2} \Rightarrow t = 1\right]$$

$$= \frac{\pi}{2}\int_0^{1} \frac{dt}{[(\sqrt{2})^2 - (t-1)^2]}\,dt$$

$$= \frac{\pi}{2}\cdot\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{2}+(t-1)}{\sqrt{2}-(t-1)}\right|_0^1$$

$$= \frac{\pi}{4\sqrt{2}}\log\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right| \quad \text{(1)}$$

26. Here, $|x-1|$, $|x-2|$ and $|x-4|$ occurs.

Now, define the absolute function as

$$|x-1| = \begin{cases} x-1, & x \ge 1 \\ -(x-1), & x < 1 \end{cases}; \quad |x-2| = \begin{cases} x-2, & x \ge 2 \\ -(x-2), & x < 2 \end{cases}$$

$$\text{and } |x-4| = \begin{cases} x-4, & x \ge 4 \\ -(x-4), & x < 4 \end{cases}$$

$$\text{Let } I = \int_1^4 (|x-1| + |x-2| + |x-4|)\,dx$$

$$= \int_1^2 (|x-1| + |x-2| + |x-4|)\,dx + \int_2^3 (|x-1| + |x-2| + |x-4|)\,dx + \int_3^4 (|x-1| + |x-2| + |x-4|)\,dx \quad \text{(1)}$$

$$= \int_1^2 \{(x-1) - (x-2) - (x-4)\}\,dx + \int_2^3 \{(x-1) + (x-2) - (x-4)\}\,dx + \int_3^4 \{(x-1) + (x-2) - (x-4)\}\,dx \quad \text{(1)}$$

$$= \int_1^2 (-x+5)\,dx + \int_2^3 (x+1)\,dx + \int_3^4 (x+1)\,dx$$

$$= \left(\frac{-x^2}{2} + 5x\right)_1^2 + \left(\frac{x^2}{2} + x\right)_2^3 + \left(\frac{x^2}{2} + x\right)_3^4 \quad \text{(1)}$$

$$= \left(\frac{-4}{2} + 10\right) - \left(\frac{-1}{2} + 5\right) + \left(\frac{9}{2} + 3\right) - \left(\frac{4}{2} + 2\right) + \left(\frac{16}{2} + 4\right) - \left(\frac{9}{2} + 3\right)$$

$$= 8 - \frac{9}{2} + \frac{15}{2} - 4 + 12 - \frac{15}{2} = 16 - \frac{9}{2} = \frac{23}{2} \quad \text{(1)}$$

27. Let $I = \int_0^{\pi} \dfrac{x\sin x}{1+\cos^2 x}\,dx$ ...(i)

$$\Rightarrow \quad I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}\,dx$$

$$\Rightarrow \quad I = \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}\,dx \quad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi} \frac{\pi\sin x}{(1+\cos^2 x)}\,dx \quad \text{(1)}$$

Put $\cos x = t \Rightarrow -\sin x\,dx = dt \Rightarrow \sin x\,dx = -dt$

when $x = 0$, then $t = 1$

and $x = \pi$, then $t = -1$

$$= -\pi\int_1^{-1} \frac{dt}{1+t^2}, \quad \text{(1)}$$

$$\Rightarrow 2I = \pi\int_{-1}^{1}\frac{dt}{(1+t^2)} = \pi[\tan^{-1} t]_{-1}^{1}$$ **(1)**

$$= \pi[\tan^{-1}1 - \tan^{-1}(-1)]$$

$$= \pi\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{\pi^2}{2} \Rightarrow I = \frac{\pi^2}{4}$$ **(1)**

28. Let $I = \int_0^{\pi}\frac{x\tan x}{\sec x + \tan x}dx$ ...(i)

$$\Rightarrow I = \int_0^{\pi}\frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)}$$

$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$\Rightarrow I = \int_0^{\pi}\frac{(\pi - x)\tan x}{\sec x + \tan x}dx$$ ...(ii) **(1)**

On adding Eqs. (i) and (ii), we get

$$\Rightarrow 2I = \int_0^{\pi}\frac{\pi\tan x}{\sec x + \tan x}dx$$

$$\Rightarrow I = \frac{\pi}{2}\int_0^{\pi}\frac{\tan x(\sec x - \tan x)}{(\sec x + \tan x)(\sec x - \tan x)}dx$$

[rationalising]

$$= \frac{\pi}{2}\int_0^{\pi}\frac{(\tan x\sec x - \tan^2 x)\,dx}{(\sec^2 x - \tan^2 x)}$$ **(1)**

$$[\because (a+b)(a-b) = a^2 - b^2]$$

$$= \frac{\pi}{2}\int_0^{\pi}\frac{\tan x\sec x - \sec^2 x + 1}{1}dx$$ **(1)**

$$[\because \tan^2 x = \sec^2 x - 1]$$

$$= \frac{\pi}{2}[\sec x - \tan x + x]_0^{\pi}$$

$$= \frac{\pi}{2}[(\sec\pi - \tan\pi + \pi) - (\sec 0 - \tan 0 + 0)]$$

$$= \frac{\pi}{2}[\sec\pi - \sec 0 - \tan\pi + \tan 0 + \pi - 0]$$

$$= \frac{\pi}{2}[-1 - 1 - 0 + 0 + \pi - 0] = \frac{\pi}{2}[\pi - 2]$$ **(1)**

29. First, define the absolute function in the given interval and then integrate it.

Let $I = \int_{-1}^{2}|x^3 - x|\,dx$

We observe that,

$$|x^3 - x| = \begin{cases} (x^3 - x), & \text{when } -1 < x < 0 \\ -(x^3 - x), & \text{when } 0 \le x < 1 \\ (x^3 - x), & \text{when } 1 \le x < 2 \end{cases}$$ **(1)**

$$\therefore I = \int_{-1}^{0}(x^3 - x)\,dx + \int_0^1 -(x^3 - x)\,dx + \int_1^2 (x^3 - x)\,dx$$ **(1)**

$$= \int_{-1}^{0}(x^3 - x)\,dx - \int_0^1 (x^3 - x)\,dx + \int_1^2 (x^3 - x)\,dx$$

$$= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} - \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_0^1 + \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_1^2$$ **(1)**

$$= \left[0 - \left(\frac{1}{4} - \frac{1}{2}\right)\right] - \left[\left(\frac{1}{4} - \frac{1}{2}\right) - 0\right] + \left[\left(\frac{16}{4} - \frac{4}{2}\right) - \left(\frac{1}{4} - \frac{1}{2}\right)\right]$$

$$= -\frac{1}{4} + \frac{1}{2} - \frac{1}{4} + \frac{1}{2} + 4 - 2 - \frac{1}{4} + \frac{1}{2}$$

$$= -\frac{3}{4} + \frac{3}{2} + 2 = \frac{-3 + 6 + 8}{4} = \frac{11}{4}$$ **(1)**

30. Let $I = \int_0^{\pi} e^{2x}\sin\left(\frac{\pi}{4} + x\right)dx$

Again, let $I_1 = \int \underset{\text{II}}{e^{2x}}\,\underset{\text{I}}{\sin\left(\frac{\pi}{4} + x\right)}dx$ ... (i)

$$= \sin\left(\frac{\pi}{4} + x\right)\int e^{2x}\,dx - \int\left\{\frac{d}{dx}\sin\left(\frac{\pi}{4} + x\right)\int e^{2x}\,dx\right\}dx$$

[using integration by parts]

$$= \sin\left(\frac{\pi}{4} + x\right)\frac{e^{2x}}{2} - \int\cos\left(\frac{\pi}{4} + x\right)\frac{e^{2x}}{2}dx$$ **(1)**

$$= \frac{e^{2x}}{2}\sin\left(\frac{\pi}{4} + x\right) - \frac{1}{2}\int \underset{\text{II}}{e^{2x}}\,\underset{\text{I}}{\cos\left(\frac{\pi}{4} + x\right)}dx$$

$$= \frac{e^{2x}}{2}\sin\left(\frac{\pi}{4} + x\right) - \frac{1}{2}\left[\cos\left(\frac{\pi}{4} + x\right)\frac{e^{2x}}{2} - \int -\sin\left(\frac{\pi}{4} + x\right)\frac{e^{2x}}{2}dx\right]$$

[using integration by parts]

$$= \frac{e^{2x}}{2}\sin\left(\frac{\pi}{4} + x\right) - \frac{e^{2x}}{4}\cos\left(\frac{\pi}{4} + x\right) - \frac{1}{4}\int e^{2x}\sin\left(\frac{\pi}{4} + x\right)dx$$ **(1)**

$$\Rightarrow I_1 = \frac{e^{2x}}{4}\left\{2\sin\left(\frac{\pi}{4} + x\right) - \cos\left(\frac{\pi}{4} + x\right)\right\} - \frac{1}{4}I_1$$

[from Eq. (i)]

$$\Rightarrow I_1 + \frac{1}{4} I_1 = \frac{e^{2x}}{4}\left\{2\sin\left(\frac{\pi}{4}+x\right) - \cos\left(\frac{\pi}{4}+x\right)\right\}$$

$$\Rightarrow \quad \frac{5}{4} I_1 = \frac{e^{2x}}{4}\left\{2\sin\left(\frac{\pi}{4}+x\right) - \cos\left(\frac{\pi}{4}+x\right)\right\}$$

$$\Rightarrow \quad I_1 = \frac{e^{2x}}{5}\left\{2\sin\left(\frac{\pi}{4}+x\right) - \cos\left(\frac{\pi}{4}+x\right)\right\} \quad \textbf{(1)}$$

$$\therefore \quad I = [I_1]_0^{\pi}$$

$$= \left[\frac{e^{2x}}{5}\left\{2\sin\left(\frac{\pi}{4}+x\right) - \cos\left(\frac{\pi}{4}+x\right)\right\}\right]_0^{\pi}$$

$$= \frac{1}{5}\left[e^{2\pi}\left\{2\sin\left(\frac{\pi}{4}+\pi\right) - \cos\left(\frac{\pi}{4}+\pi\right)\right\} - e^0\left\{2\sin\left(\frac{\pi}{4}+0\right) - \cos\left(\frac{\pi}{4}+0\right)\right\}\right]$$

$$= \frac{1}{5}\left[e^{2\pi}\left\{-2\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right\} - e^0\left\{2\sin\frac{\pi}{4} - \cos\frac{\pi}{4}\right\}\right]$$

$$= \frac{1}{5}\left[e^{2\pi}\left\{-2\times\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right\} - 1\left\{2\times\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right\}\right]$$

$$= \frac{1}{5}\left[e^{2\pi}\left\{-\frac{1}{\sqrt{2}}\right\} - \frac{1}{\sqrt{2}}\right] = -\frac{1}{5\sqrt{2}}[e^{2\pi}+1] \quad \textbf{(1)}$$

31. Let $I = \int_{-2}^{2} \frac{x^2}{1+5^x}\,dx$ ...(i)

$$= \int_{-2}^{2} \frac{(2-2-x)^2}{1+5^{2-2-x}}\,dx$$

$$\left[\because \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx\right]$$

$$= \int_{-2}^{2} \frac{x^2}{1+5^{-x}}\,dx \quad \textbf{(1)}$$

$$\Rightarrow I = \int_{-2}^{2} \frac{5^x}{5^x+1}\, x^2\,dx \quad \text{...(ii)} \ \textbf{(1/2)}$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_{-2}^{2}\left(\frac{1+5^x}{5^x+1}\right) x^2\,dx = \int_{-2}^{2} x^2\,dx \quad \textbf{(1)}$$

$$\Rightarrow \quad 2I = 2\int_0^2 x^2\,dx$$

$$\left[\because x^2 \text{ is even, so } \int_{-2}^{2} x^2\,dx = 2\int_0^2 x^2\,dx\right] \ \textbf{(1/2)}$$

$$\Rightarrow \quad I = \left[\frac{x^3}{3}\right]_0^2 = \frac{1}{3}(2^3 - 0) = \frac{8}{3} \quad \textbf{(1)}$$

32. Let $I = \int_0^{3/2} |x\cos\pi x|\,dx$

consider, $x\cos\pi x = 0$

$\Rightarrow \quad x = 0$ or $\cos\pi x = 0$

$\Rightarrow \quad x = 0$ or $\pi x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$

$\left[\because \cos\frac{n\pi}{2} = 0, n \text{ being an odd integer}\right]$

$\Rightarrow \quad x = 0$ or $x = \frac{1}{2}, \frac{3}{2}$ $\qquad \left[\because 0 < x < \frac{3}{2}\right]$

$\Rightarrow \quad x = 0, \frac{1}{2}, \frac{3}{2}$

So, let us divide the integral at $x = \frac{1}{2}$ **(1)**

Note that $|x\cos\pi x| = \begin{cases} x\cos\pi x, & 0 \le x \le \frac{1}{2} \\ -x\cos\pi x, & \frac{1}{2} \le x \le \frac{3}{2} \end{cases}$

$\left[\because 0 \le x \le \frac{1}{2} \Rightarrow 0 \le \pi x \le \frac{\pi}{2} \Rightarrow \cos\pi x \ge 0 \text{ and } \frac{1}{2} \le x \le \frac{3}{2} \Rightarrow \frac{\pi}{2} \le \pi x \le \frac{3\pi}{2} \Rightarrow \cos\pi x \le 0\right]$

Now, $I = \int_0^{1/2} |x\cos\pi x|\,dx + \int_{1/2}^{3/2} |x\cos\pi x|\,dx$

$$= \int_0^{1/2} x\cos\pi x\,dx - \int_{1/2}^{3/2} x\cos\pi x\,dx \quad \text{...(i)} \ \textbf{(1)}$$

Let $I_1 = \int \underset{\text{I}}{x}\,\underset{\text{II}}{\cos\pi x}\,dx = x\frac{\sin\pi x}{\pi} - \int \frac{\sin\pi x}{\pi}\,dx$

[using integration by parts]

$$= \frac{x\sin\pi x}{\pi} + \frac{\cos\pi x}{\pi^2} \quad \textbf{(1)}$$

Now, From Eq. (i), we have

$$I = \left[x\frac{\sin\pi x}{\pi} + \frac{\cos\pi x}{\pi^2}\right]_0^{1/2} - \left[\frac{x\sin\pi x}{\pi} + \frac{\cos\pi x}{\pi^2}\right]_{1/2}^{3/2}$$

$$= \left[\left(\frac{1}{2\pi}\sin\frac{\pi}{2} + \frac{1}{\pi^2}\cos\frac{\pi}{2}\right) - \left(0 + \frac{\cos 0}{\pi^2}\right)\right] - \left[\left(\frac{3}{2\pi}\sin\frac{3\pi}{2} + \frac{1}{\pi^2}\cos\frac{3\pi}{2}\right) - \left(\frac{1}{2\pi}\sin\frac{\pi}{2} + \frac{1}{\pi^2}\cos\frac{\pi}{2}\right)\right]$$

$$= \frac{1}{2\pi} - \frac{1}{\pi^2} - \left(\frac{-3}{2\pi} - \frac{1}{2\pi}\right)$$

$$= \frac{1}{2\pi} - \frac{1}{\pi^2} + \frac{3}{2\pi} + \frac{1}{2\pi} = \frac{5}{2\pi} - \frac{1}{\pi^2} = \frac{5\pi - 2}{2\pi^2} \quad \textbf{(1)}$$

33. Let $I = \int_0^\pi \frac{x}{1+\sin\alpha \sin x}dx$ ...(i)

$$= \int_0^\pi \frac{(\pi - x)}{1+\sin\alpha\sin(\pi - x)}dx$$

$$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$= \int_0^\pi \frac{(\pi - x)}{1+\sin\alpha \sin x}dx \quad ...(ii)$$

$$[\because \sin(\pi - x) = \sin x]$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^\pi \frac{\pi}{1+\sin\alpha\sin x}dx = \pi\int_0^\pi \frac{dx}{1+\sin\alpha\sin x} \quad (1)$$

$$= \pi\int_0^\pi \frac{dx}{1+\sin\alpha\sin x}$$

$$= \pi\int_0^\pi \frac{dx}{1+\sin\alpha\left(\frac{2\tan x/2}{1+\tan^2 x/2}\right)} \quad (1/2)$$

$$= \pi\int_0^\pi \frac{(1+\tan^2 x/2)dx}{(1+\tan^2 x/2)+2\sin\alpha\tan x/2}$$

$$= \pi\int_0^\pi \frac{(\sec^2 x/2)dx}{\tan^2 x/2 + 2\sin\alpha\cdot\tan x/2 + 1}$$

$\therefore$ Put $\tan\frac{x}{2} = t \Rightarrow \sec^2\frac{x}{2}\cdot\frac{1}{2}dx = dt$

$\Rightarrow$ $\sec^2\frac{x}{2}dx = 2dt$

Also, when $x = 0$, then $t = 0$ and when $x = \pi$, then $t \to \infty$

$$\therefore \quad 2I = \pi\int_0^\infty \frac{2dt}{t^2 + 2\sin\alpha\cdot t + 1} \quad (1)$$

$$\Rightarrow \quad I = \pi\int_0^\infty \frac{dt}{t^2 + 2\sin\alpha\cdot t + \sin^2\alpha + \cos^2\alpha}$$

$$[\because \sin^2\alpha + \cos^2\alpha = 1] \quad (1/2)$$

$$= \pi\int_0^\infty \frac{dt}{(t+\sin\alpha)^2 + (\cos\alpha)^2}$$

$$= \frac{\pi}{\cos\alpha}\left[\tan^{-1}\left(\frac{t+\sin\alpha}{\cos\alpha}\right)\right]_0^\infty$$

$$\left[\because \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C\right]$$

$$= \frac{\pi}{\cos\alpha}[\tan^{-1}(\infty) - \tan^{-1}(\tan\alpha)]$$

$$= \frac{\pi}{\cos\alpha}\left[\frac{\pi}{2} - \alpha\right] \quad (1)$$

34. Let $I = \int_{-\pi}^{\pi} (\cos ax - \sin bx)^2 dx$

$$= \int_{-\pi}^{\pi} (\cos^2 ax + \sin^2 bx - 2\cos ax \sin bx)dx$$

$$= \int_{-\pi}^{\pi} (\cos^2 ax + \sin^2 bx)dx - 2\int_{-\pi}^{\pi} \cos ax \sin bx\, dx$$

$$= I_1 - I_2 \quad (1)$$

Now consider,

$$I_1 = \int_{-\pi}^{\pi} (\cos^2 ax + \sin^2 bx)dx \quad \text{[be an even function]}$$

$$= 2\int_0^\pi (\cos^2 ax + \sin^2 bx)dx$$

$$= 2\int_0^\pi \left(\frac{1+\cos 2ax}{2} + \frac{1-\cos 2bx}{2}\right)dx \quad (1)$$

$$= \int_0^\pi (1 + \cos 2ax + 1 - \cos 2bx)dx$$

$$= \int_0^\pi (2 + \cos 2ax - \cos 2bx)dx$$

$$= \left(2x + \frac{\sin 2ax}{2a} - \frac{\sin 2bx}{2b}\right)_0^\pi$$

$$= \left(2\pi + \frac{\sin 2a\pi}{2a} - \frac{\sin 2b\pi}{2b}\right) - 0$$

$$= 2\pi + \frac{\sin 2a\pi}{2a} - \frac{\sin 2b\pi}{2b}$$

and $I_2 = 2\int_{-\pi}^{\pi} (\cos ax \sin bx)dx$ [be an odd function]

$$= 0 \quad \left[\because \int_{-a}^{a} f(x) = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(x) \text{ is even} \\ 0, & \text{if } f(x) \text{ is odd}\end{cases}\right] \quad (1)$$

$$\therefore I = I_1 - I_2 = 2\pi + \frac{\sin 2a\pi}{2a} - \frac{\sin 2b\pi}{2b} \quad (1)$$

35. Let $I = \int_0^{\pi/4} \frac{dx}{\cos^3 x\sqrt{2\sin 2x}}$

$$= \int_0^{\pi/4} \frac{dx}{\cos^3 x\sqrt{2(2\sin x\cos x)}}$$

$$[\because \sin 2x = 2\sin x\cos x]$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{dx}{\cos^3 x\cos^{1/2} x\sin^{1/2} x}$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{dx}{\cos^{7/2} x\sin^{1/2} x}$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{\sec^4 x}{\cos^{\frac{7}{2}-4} x\sin^{1/2} x}dx$$

[dividing numerator and denominator by $\cos^4 x$]

$$= \frac{1}{2}\int_0^{\pi/4} \frac{\sec^4 x}{\cos^{-1/2} x\sin^{1/2} x}dx$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{\sec^2 x(1+\tan^2 x)}{\tan^{1/2} x}dx$$

$$[\because \sec^2 x - \tan^2 x = 1] \quad (1)$$

Put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

Lower limit when $x = 0$, then $t = \tan 0 = 0$

Upper limit when $x = \frac{\pi}{4}$, then $t = \tan\frac{\pi}{4} = 1$

$$\therefore \quad I=\frac{1}{2}\int_0^1\left(\frac{1+t^2}{t^{1/2}}\right)dt \quad \textbf{(1)}$$

$$=\frac{1}{2}\int_0^1(t^{-1/2}+t^{3/2})\,dt \quad \textbf{(1)}$$

$$=\frac{1}{2}\left[2t^{1/2}+\frac{2}{5}t^{5/2}\right]_0^1=\left[t^{1/2}+\frac{1}{5}t^{5/2}\right]_0^1$$

$$=(1)^{1/2}+\frac{1}{5}(1)^{5/2}-0=1+\frac{1}{5}=\frac{6}{5} \quad \textbf{(1)}$$

36. Let $I=\int_{-\pi/2}^{\pi/2}\frac{\cos x}{1+e^x}dx$

$$=\int_{-\pi/2}^{0}\frac{\cos x}{1+e^x}dx+\int_0^{\pi/2}\frac{\cos x}{1+e^x}dx.$$

In first integral, put $x=-t \Rightarrow dx=-dt$

Upper limit when $x=0$, then $t=0$

Lower limit when $x=-\frac{\pi}{2}$, then $t=\frac{\pi}{2}$. **(1)**

$$\therefore I=\int_{\pi/2}^{0}\frac{\cos t}{1+e^{-t}}(-dt)+\int_0^{\pi/2}\frac{\cos x}{1+e^x}dx$$

$$=\int_0^{\pi/2}\frac{\cos t}{1+\frac{1}{e^t}}dt+\int_0^{\pi/2}\frac{\cos x}{1+e^x}dx$$

$$\left[\because -\int_a^0 f(x)dx=\int_0^a f(x)dx\right]$$

$$=\int_0^{\pi/2}\frac{e^t\cos t}{e^t+1}dt+\int_0^{\pi/2}\frac{\cos x}{1+e^x}dx$$

$$=\int_0^{\pi/2}\frac{e^x\cos x}{e^x+1}dx+\int_0^{\pi/2}\frac{\cos x}{1+e^x}dx \quad \textbf{(1)}$$

$$\left[\because \int_a^b f(x)dx=\int_a^b f(t)dt\right]$$

$$=\int_0^{\pi/2}\frac{(1+e^x)\cos x}{(e^x+1)}dx=\int_0^{\pi/2}\cos x\,dx \quad \textbf{(1)}$$

$$=[\sin x]_0^{\pi/2}=\sin\frac{\pi}{2}-0=1 \quad \textbf{(1)}$$

37. Let $I=\int_0^{\pi/4}\log(1+\tan x)\,dx$

$$\therefore \quad I=\int_0^{\pi/4}\log\left[1+\tan\left(\frac{\pi}{4}-x\right)\right]dx$$

$$\left[\because \int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$$

$$=\int_0^{\pi/4}\log\left(1+\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right)dx$$

$$\left[\because \tan(x-y)=\frac{\tan x-\tan y}{1+\tan x\tan y}\right] \quad \textbf{(1)}$$

$$=\int_0^{\pi/4}\log\left(1+\frac{1-\tan x}{1+\tan x}\right)dx$$

$$=\int_0^{\pi/4}\log\left(\frac{2}{1+\tan x}\right)dx$$

$$=\int_0^{\pi/4}(\log 2)\,dx-\int_0^{\pi/4}\log(1+\tan x)\,dx \quad \textbf{(1)}$$

$$\left[\because \log\frac{m}{n}=\log m-\log n\right]$$

$$=\int_0^{\pi/4}(\log 2)\,dx-I$$

$$\Rightarrow 2I=\int_0^{\pi/4}\log 2\,dx=\log 2[x]_0^{\pi/4}=\log 2\left[\frac{\pi}{4}-0\right] \quad \textbf{(1)}$$

$$\therefore \quad I=\frac{\pi}{8}\log 2 \quad \textbf{(1)}$$

38. Let $I=\int_{\pi/6}^{\pi/3}\frac{\sin x+\cos x}{\sqrt{\sin 2x}}dx$

Put $\quad \sin x-\cos x=t$

$\Rightarrow \quad (\cos x+\sin x)\,dx=dt$ **(1/2)**

Lower limit when $x=\frac{\pi}{6}$, then

$$t=\sin\frac{\pi}{6}-\cos\frac{\pi}{6}=\frac{1}{2}-\frac{\sqrt{3}}{2}=\frac{1-\sqrt{3}}{2}$$

and **Upper limit** when $x=\frac{\pi}{3}$, then

$$t=\sin\frac{\pi}{3}-\cos\frac{\pi}{3}=\frac{\sqrt{3}}{2}-\frac{1}{2}=\frac{\sqrt{3}-1}{2}. \quad \textbf{(1)}$$

Also, $\quad (\sin x-\cos x)^2=t^2$

$\Rightarrow \quad \sin^2 x+\cos^2 x-2\sin x\cos x=t^2$

$\Rightarrow \quad 1-\sin 2x=t^2$

$\Rightarrow \quad \sin 2x=1-t^2$ **(1/2)**

$$\therefore \quad I=\int_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}}\frac{dt}{\sqrt{1-t^2}}=[\sin^{-1}t]_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}} \quad \textbf{(1)}$$

$$=\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)-\sin^{-1}\left(\frac{1-\sqrt{3}}{2}\right)$$

$$=\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)+\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$$

$$[\because \sin^{-1}(-x)=-\sin^{-1}x]$$

$$=2\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right) \quad \textbf{(1)}$$

39. Let $I=\int_0^{\pi/2}x^2\sin x\,dx$

Now, $\int \underset{\text{I}}{x^2}\underset{\text{II}}{\sin x}\,dx=-x^2\cos x+2\int \underset{\text{I}}{x}\underset{\text{II}}{\cos x}\,dx$

$$=-x^2\cos x+2\left[x(\sin x)-\int 1\cdot(\sin x)\,dx\right]$$

[using integration by parts]

$$=-x^2\cos x+2(x\sin x+\cos x) \quad \textbf{(1)}$$

$$\therefore I = \int_0^{\pi/2} x^2 \sin x\,dx$$

$$= [-x^2\cos x + 2(x\sin x + \cos x)]_0^{\pi/2} \quad \textbf{(1)}$$

$$= \left[-\left(\frac{\pi}{2}\right)^2 \cos\left(\frac{\pi}{2}\right) + 2\left(\frac{\pi}{2}\sin\frac{\pi}{2} + \cos\frac{\pi}{2}\right) - 2(0+\cos 0)\right] \quad \textbf{(1)}$$

$$= -\frac{\pi^2}{4}\times 0 + 2\left(\frac{\pi}{2}+0\right) - 2(0+1) = \pi - 2 \quad \textbf{(1)}$$

40. Let $\quad I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x}\,dx \quad \ldots(\text{i})$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\sin^2\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}\,dx$$

$$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\cos^2 x}{\cos x + \sin x}\,dx \quad \ldots(\text{ii}) \ \textbf{(1)}$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\sin^2 x + \cos^2 x}{\sin x + \cos x}\,dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \frac{1}{\sin x + \cos x}\,dx$$

$$[\because \sin^2 x + \cos^2 x = 1]$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \frac{1}{\frac{2\tan x/2}{1+\tan^2 x/2} + \frac{1-\tan^2 x/2}{1+\tan^2 x/2}}\,dx$$

$$\left[\because \sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} \text{ and } \cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right]$$

$$= \int_0^{\pi/2} \frac{\sec^2 x/2}{2\tan x/2 + 1 - \tan^2 x/2}\,dx$$

Put $\tan\frac{x}{2} = t \Rightarrow \sec^2\frac{x}{2}\cdot\frac{1}{2}dx = dt \Rightarrow \sec^2\frac{x}{2}dx = 2dt$

Lower limit when $x = 0$, then $t = \tan 0 = 0$

Upper limit when $x = \frac{\pi}{2}$, then $t = \tan\frac{\pi}{4} = 1$. **(1)**

$$\therefore \quad 2I = \int_0^1 \frac{2dt}{2t+1-t^2} = 2\int_0^1 \frac{dt}{-[t^2-2t-1]}$$

$$= 2\int_0^1 \frac{dt}{-[(t-1)^2-1-1]} = 2\int_0^1 \frac{dt}{(\sqrt{2})^2-(t-1)^2}$$

$$= \left[\frac{2}{2\sqrt{2}}\log\left|\frac{\sqrt{2}+t-1}{\sqrt{2}-t+1}\right|\right]_0^1$$

$$\left[\because \int\frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C\right]$$

$$= \frac{1}{\sqrt{2}}\left[\log\frac{\sqrt{2}+1-1}{\sqrt{2}-1+1} - \log\frac{\sqrt{2}+0-1}{\sqrt{2}-0+1}\right]$$

$$= \frac{1}{\sqrt{2}}\left[\log 1 - \log\frac{\sqrt{2}-1}{\sqrt{2}+1}\right] \quad \textbf{(1)}$$

$$= -\frac{1}{\sqrt{2}}\log\left[\frac{\sqrt{2}-1}{\sqrt{2}+1}\times\frac{\sqrt{2}+1}{\sqrt{2}+1}\right] \quad [\because \log 1 = 0]$$

$$= \frac{-1}{\sqrt{2}}\log\frac{2-1}{(\sqrt{2}+1)^2} \quad [\because (a-b)(a+b) = a^2-b^2]$$

$$= \frac{-1}{\sqrt{2}}\log\frac{1}{(\sqrt{2}+1)^2}$$

$$\Rightarrow \quad 2I = \frac{2}{\sqrt{2}}\log(\sqrt{2}+1) \Rightarrow I = \frac{1}{\sqrt{2}}\log(\sqrt{2}+1) \ \textbf{(1)}$$

Hence proved.

41. First, we redefined the integrand of the integral between the given limits (2, 5). After that integrate and simplify it.

For, $2 \le x < 5,\ |x-2| = (x-2)$

$2 \le x < 3,\ |x-3| = -(x-3)$

$3 \le x < 5.\ |x-3| = (x-3)$

and $2 \le x < 5,\ |x-5| = (5-x)$ **(1)**

$$\therefore \quad I = \int_2^5 [\,|x-2| + |x-3| + |x-5|\,]\,dx$$

$$= \int_2^5 (x-2)dx + \int_2^3 (3-x)\,dx + \int_3^5 (x-3)dx + \int_2^5 (5-x)\,dx$$

$$= \left[\frac{x^2}{2}-2x\right]_2^5 + \left[3x-\frac{x^2}{2}\right]_2^3 + \left[\frac{x^2}{2}-3x\right]_3^5 + \left[5x-\frac{x^2}{2}\right]_2^5 \quad \textbf{(1)}$$

$$= \left[\left(\frac{25}{2}-10\right)-(2-4)\right] + \left[\left(9-\frac{9}{2}\right)-(6-2)\right] + \left[\left(\frac{25}{2}-15\right)-\left(\frac{9}{2}-9\right)\right] + \left[\left(25-\frac{25}{2}\right)-(10-2)\right] \quad \textbf{(1)}$$

$$= \left[\frac{25}{2}-8\right] + \left[\frac{9}{2}-4\right] + [8-6] + \left[\frac{25}{2}-8\right]$$

$$= \frac{9}{2} + \frac{1}{2} + 2 + \frac{9}{2} = \frac{19}{2} + 2 = \frac{23}{2} \quad \textbf{(1)}$$

42. Let $I = \int_0^4 [\,|x| + |x-2| + |x-4|\,]\,dx$

Here, redefined the given integrand in given interval (0, 4).

For, $0 < x < 4,\ |x| = x$

$0 < x \le 2,\ |x-2| = -(x-2)$

$2 \le x < 4,\ |x-2| = (x-2)$

$0 < x < 4,\ |x-4| = -(x-4)$ **(1)**

$\therefore \quad I = \int_0^4 x\,dx + \int_0^2 (2-x)dx + \int_2^4 (x-2)\,dx + \int_0^4 (4-x)\,dx$ **(1)**

$$= \left[\frac{x^2}{2}\right]_0^4 + \left[2x - \frac{x^2}{2}\right]_0^2 + \left[\frac{x^2}{2} - 2x\right]_2^4 + \left[4x - \frac{x^2}{2}\right]_0^4 \quad \textbf{(1)}$$

$$= (8) + [(4-2)-0] + [(8-8)-(2-4)] + \left[16 - \frac{16}{2}\right]$$

$$= 8 + 2 + 2 + (16-8) = 20 \quad \textbf{(1)}$$

43. Do same as Q. No. 41. **[Ans. 5]**

44. Let $I = \int_0^{2\pi} \frac{dx}{1 + e^{\sin x}}$ ...(i)

$$\Rightarrow I = \int_0^{2\pi} \frac{dx}{1 + e^{\sin(2\pi - x)}} \quad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)dx\right]$$

(1)

$$\Rightarrow I = \int_0^{2\pi} \frac{dx}{1 + e^{-\sin x}} = \int_0^{2\pi} \frac{e^{\sin x}}{e^{\sin x} + 1} dx \quad ...(ii)$$

On adding Eqs. (i) and (ii), we get

$$I + I = \int_0^{2\pi} \frac{dx}{1 + e^{\sin x}} + \int_0^{2\pi} \frac{e^{\sin x}}{1 + e^{\sin x}} dx \quad \textbf{(1)}$$

$$\Rightarrow \quad 2I = \int_0^{2\pi} \frac{(1 + e^{\sin x})}{(1 + e^{\sin x})} dx$$

$$\Rightarrow \quad 2I = \int_0^{2\pi} 1\,dx = [x]_0^{2\pi} = 2\pi - 0$$

$$\therefore \quad I = \pi \quad \textbf{(2)}$$

45. Here, the power of numerator is greater than the power of denominator. So, first we add and subtract 1 in numerator and use formula $(a^2 - b^2) = (a-b)(a+b)$ to simplify it and then integrate it.

Let $I = \int_0^1 \frac{x^4 + 1}{x^2 + 1} dx \Rightarrow I = \int_0^1 \frac{(x^4 - 1) + 2}{x^2 + 1} dx$

$$= \int_0^1 \frac{(x^2 - 1)(x^2 + 1) + 2}{x^2 + 1} dx$$

$$[\because (a^2 - b^2) = (a-b)(a+b)]$$

$$= \int_0^1 \left[\frac{(x^2 - 1)(x^2 + 1)}{x^2 + 1} + \frac{2}{x^2 + 1}\right] dx \quad \textbf{(1)}$$

$$\Rightarrow I = \int_0^1 \left[x^2 - 1 + \frac{2}{x^2 + 1}\right] dx$$

$$\Rightarrow I = \left[\frac{x^3}{3} - x + 2\tan^{-1} x\right]_0^1 \quad \textbf{(1)}$$

$$\therefore I = \frac{1}{3} - 1 + 2\tan^{-1} 1 - 0 = -\frac{2}{3} + 2 \times \frac{\pi}{4} = \frac{3\pi - 4}{6}$$

(2)

46. Let $I = \int_0^{\pi/2} \frac{x + \sin x}{1 + \cos x} dx$

$$\Rightarrow I = \int_0^{\pi/2} \frac{x + 2\sin\frac{x}{2}\cdot\cos\frac{x}{2}}{2\cos^2\frac{x}{2}} dx \quad \textbf{(1)}$$

$$\left[\begin{array}{l}\because \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} \\ \text{and } 1 + \cos x = 2\cos^2\frac{x}{2}\end{array}\right]$$

$$\Rightarrow I = \frac{1}{2}\int_0^{\pi/2} \underset{\text{I}}{x}\,\underset{\text{II}}{\sec^2\frac{x}{2}}\, dx + \int_0^{\pi/2} \tan\frac{x}{2} dx \quad \textbf{(1)}$$

$$\Rightarrow I = \frac{1}{2}\left\{\left[x\int \sec^2\frac{x}{2} dx\right]_0^{\pi/2} - \int_0^{\pi/2}\left[\frac{d}{dx}(x)\int\left(\sec^2\frac{x}{2}dx\right)\right]dx\right\} + \int_0^{\pi/2}\tan\frac{x}{2}dx$$

$$\Rightarrow I = \frac{1}{2}\left\{\left[x\cdot\frac{\tan\frac{x}{2}}{\frac{1}{2}}\right]_0^{\pi/2} - \int_0^{\pi/2}\frac{\tan\frac{x}{2}}{\frac{1}{2}}dx\right\} + \int_0^{\pi/2}\tan\frac{x}{2}dx$$

[using integration by parts]

$$= \left[x\cdot\tan\frac{x}{2}\right]_0^{\pi/2} - \int_0^{\pi/2}\tan\frac{x}{2}dx + \int_0^{\pi/2}\tan\frac{x}{2}dx \quad \textbf{(1)}$$

$$= \frac{\pi}{2}\cdot\tan\frac{\pi}{4} - 0$$

$$\therefore \quad I = \frac{\pi}{2} \quad \left[\because \tan\frac{\pi}{4} = 1\right] \quad \textbf{(1)}$$

47. Here, the power of numerator and denominator are same. So, first we divide numerator by denominator and write integrand in the form $\left(\frac{R}{D} + Q\right)$, where R = remainder, Q = quotient and D = divisor. Now, integrate it easily by using partial fraction.

Let $\quad I = \int_1^2 \frac{5x^2}{x^2 + 4x + 3} dx$

$$I = 5\int_1^2\left(1 + \frac{-4x - 3}{x^2 + 4x + 3}\right)dx$$

$$= 5\int_1^2 dx - 5\int_1^2 \frac{4x + 3}{x^2 + 4x + 3} dx$$

$$\Rightarrow \quad I = 5[x]_1^2 - 5\int_1^2 \frac{4x + 3}{(x+3)(x+1)} dx \quad ...(i)\ \textbf{(1)}$$

By using partial fraction,

let $\dfrac{4x+3}{(x+3)(x+1)} = \dfrac{A}{x+3} + \dfrac{B}{x+1}$

$\Rightarrow \dfrac{4x+3}{(x+3)(x+1)} = \dfrac{A(x+1)+B(x+3)}{(x+3)(x+1)}$

$\Rightarrow \quad 4x+3 = A(x+1)+B(x+3)$

On comparing the coefficients of like terms from both sides, we get

$A+B=4 \Rightarrow A=4-B$

and $\quad A+3B=3 \Rightarrow 4-B+3B=3$

$\Rightarrow \quad B=-\dfrac{1}{2}$, then $A=4+\dfrac{1}{2}=\dfrac{9}{2}$ **(1)**

Now, from Eq. (i), we get

$$I = 5(2-1) - 5\int_1^2 \left(\frac{9/2}{x+3} + \frac{-1/2}{x+1}\right) dx$$

$$= 5 - 5\left[\frac{9}{2}\log|x+3| - \frac{1}{2}\log|x+1|\right]_1^2 \quad \textbf{(1)}$$

$$= 5 - 5\left[\left(\frac{9}{2}\log 5 - \frac{1}{2}\log 3\right) - \left(\frac{9}{2}\log 4 - \frac{1}{2}\log 2\right)\right]$$

$$= 5 - 5\left[\frac{9}{2}(\log 5 - \log 4) - \frac{1}{2}(\log 3 - \log 2)\right]$$

$$= 5 - 5\left[\frac{9}{2}\log\frac{5}{4} - \frac{1}{2}\log\frac{3}{2}\right]$$

$$\left[\because \log m - \log n = \log\frac{m}{n}\right]$$

$$= 5 - \frac{45}{2}\log\frac{5}{4} + \frac{5}{2}\log\frac{3}{2} \quad \textbf{(1)}$$

48. Let $I = \displaystyle\int_0^1 \frac{\log|1+x|}{1+x^2} dx$

Put $\quad x = \tan\theta$

$\Rightarrow \quad dx = \sec^2\theta \, d\theta$

Lower limit when $x=0$, then $\theta = \tan^{-1} 0 = 0$

Upper limit when $x=1$, then $\theta = \tan^{-1} 1 = \dfrac{\pi}{4}$.

$$\therefore \quad I = \int_0^{\pi/4} \frac{\log|1+\tan\theta|}{1+\tan^2\theta} \sec^2\theta \, d\theta \quad \textbf{(1)}$$

$$\Rightarrow \quad I = \int_0^{\pi/4} \frac{\log|1+\tan\theta|\sec^2\theta}{\sec^2\theta} d\theta$$

$$\Rightarrow \quad I = \int_0^{\pi/4} \log|1+\tan\theta| \, d\theta \quad \text{...(i)}$$

$$= \int_0^{\pi/4} \log\left|1+\tan\left(\frac{\pi}{4}-\theta\right)\right| d\theta$$

$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right] \textbf{(1)}$$

$$= \int_0^{\pi/4} \log\left|1+\frac{1-\tan\theta}{1+\tan\theta}\right| d\theta$$

$$= \int_0^{\pi/4} \log\left|\frac{2}{1+\tan\theta}\right| d\theta$$

$$= \int_0^{\pi/4} \log 2 \, d\theta - \int_0^{\pi/4} \log|1+\tan\theta| \, d\theta \quad \textbf{(1)}$$

$\Rightarrow \quad I = \log 2 \cdot [\theta]_0^{\pi/4} - I \quad$ [from Eq. (i)]

$\Rightarrow \quad 2I = \dfrac{\pi}{4}\log 2$

$\therefore \quad I = \dfrac{\pi}{8}\log 2$ **(1)**

49. Let $I = \displaystyle\int_0^1 \log\left|\frac{1}{x}-1\right| dx$

$$\Rightarrow \quad I = \int_0^1 \log\left|\frac{1-x}{x}\right| dx \quad \text{...(i)}$$

$$\Rightarrow \quad I = \int_0^1 \log\left|\frac{1-(1-x)}{1-x}\right| dx \quad \textbf{(1)}$$

$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$\Rightarrow \quad I = \int_0^1 \log\left|\frac{x}{1-x}\right| dx \quad \text{...(ii)} \textbf{ (1)}$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^1 \log\left|\frac{1-x}{x}\right| dx + \int_0^1 \log\left|\frac{x}{1-x}\right| dx$$

$$\Rightarrow \quad 2I = \int_0^1 \left[\log\left|\frac{1-x}{x}\right| + \log\left|\frac{x}{1-x}\right| dx\right] \quad \textbf{(1)}$$

$$\Rightarrow \quad 2I = \int_0^1 \log\left|\left(\frac{1-x}{x} \times \frac{x}{1-x}\right)\right| dx$$

$$[\because \log m + \log n = \log(m \times n)]$$

$$\Rightarrow \quad 2I = \int_0^1 \log 1 \, dx$$

$$\Rightarrow \quad 2I = \int_0^1 0 \, dx = 0 \quad [\because \log 1 = 0]$$

$\therefore \quad I = 0$ **(1)**

50. Let $I = \displaystyle\int_0^\pi \frac{x}{1+\sin x} dx \quad$ ...(i)

$$\therefore \quad I = \int_0^\pi \frac{\pi - x}{1+\sin(\pi - x)} dx$$

$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$\Rightarrow \quad I = \int_0^{\pi} \frac{\pi - x}{1 + \sin x}\,dx$$

$[\because \sin(\pi - x) = \sin x]$ ...(ii) (1)

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi} \frac{x + \pi - x}{1 + \sin x}\,dx \quad (1)$$

$$= \pi \int_0^{\pi} \frac{1}{1 + \sin x}\,dx$$

$$= \pi \int_0^{\pi} \frac{1}{1 + \sin x} \times \frac{1 - \sin x}{1 - \sin x}\,dx \quad (1)$$

$$= \pi \int_0^{\pi} \frac{1 - \sin x}{1 - \sin^2 x}\,dx = \pi \int_0^{\pi} \frac{1 - \sin x}{\cos^2 x}\,dx$$

$$= \pi \int_0^{\pi} (\sec^2 x - \tan x \sec x)\,dx$$

$$= \pi[\tan x - \sec x]_0^{\pi}$$

$$= \pi[(\tan\pi - \sec\pi) - (\tan 0 - \sec 0)]$$

$$= \pi[(0 - (-1)) - (0 - 1)] = \pi(1 + 1)$$

$$\Rightarrow 2I = 2\pi \quad \Rightarrow I = \pi \quad (1)$$

51. We have, $\int_1^3 (x^2 + 2 + e^{2x})\,dx$

On comparing with $\int_a^b f(x)\,dx$, we get

$$a = 1, b = 3, nh = 3 - 1 = 2, f(x) = x^2 + 2 + e^{2x}$$

Clearly, $f(1) = 1^2 + 2 + e^{2 \times 1} = 3 + e^2$

$$f(1 + h) = (1 + h)^2 + 2 + e^{2(1 + h)}$$

$$= 1 + 2h + h^2 + 2 + e^{2 + 2h}$$

$$= 3 + 2h + h^2 + e^2 \cdot e^{2h}$$

$$f(1 + 2h) = (1 + 2h)^2 + 2 + e^{2(1 + 2h)}$$

$$= 1 + 4h + 4h^2 + 2 + e^{2 + 4h}$$

$$= 3 + 4h + 4h^2 + e^2 \cdot e^{4h}$$

$$f(1 + (n - 1)h) = (1 + (n - 1)h)^2 + 2 + e^{2(1 + (n-1)h)}$$

$$= 1 + 2(n - 1)h + (n - 1)^2 h^2 + 2 + e^{2 + 2(n-1)h}$$

$$= 3 + 2(n - 1)h + (n - 1)^2 h^2 + e^2 \cdot e^{2(n-1)h}$$

$$\therefore f(1) + f(1 + h) + f(1 + 2h) + \ldots + f(1 + (n - 1)h)$$

$$= (3 + 3 + 3 + \ldots + 3) + 2h(1 + 2 + 3 + \ldots + (n - 1)) + h^2(1^2 + 2^2 + 3^2 + \ldots + (n - 1)^2) + e^2(1 + e^{2h} + \ldots + e^{2h(n-1)})$$

$$= 3n + \frac{2h(n)(n - 1)}{2} + h^2 \frac{n(n - 1)(2n - 1)}{6} + e^2\left[\frac{e^{2nh} - 1}{e^{2h} - 1}\right]$$

$$= 3n + nh(n - 1) + \frac{h^2 n(n - 1)(2n - 1)}{6} + e^2\left(\frac{e^{2nh} - 1}{e^{2h} - 1}\right)$$

Now, $\int_1^3 (x^2 + 2 + e^{2x})\,dx$

$$= \lim_{h \to 0} h\{f(1) + f(1 + h) + f(1 + 2h) + \ldots + f(1 + (n - 1)h)\}$$

$$= \lim_{h \to 0} h\left[3n + nh(n - 1) + \frac{h^2 n(n - 1)(2n - 1)}{6} + e^2\left(\frac{e^{2nh} - 1}{e^{2h} - 1}\right)\right]$$

$$= \lim_{h \to 0}\left[3nh + nh(nh - h) + \frac{nh(nh - h)(2nh - h)}{6} + \frac{e^2(e^{2nh} - 1)}{\frac{2(e^{2h} - 1)}{2h}}\right]$$

$$= 3 \times 2 + 2(2 - 0) + \frac{2(2 - 0)(4 - 0)}{6} + \frac{e^2(e^4 - 1)}{2}$$

$$= 6 + 4 + \frac{8}{3} + \frac{e^6 - e^2}{2} = \frac{38}{3} + \frac{e^6 - e^2}{2}$$

52. Let $I = \int_0^{\pi/4} \frac{\sin x + \cos x}{16 + 9\sin 2x}\,dx$

$$= \int_0^{\pi/4} \frac{\sin x + \cos x}{9(2\sin x \cos x) + 16}\,dx$$

$$= \int_0^{\pi/4} \frac{\sin x + \cos x}{-9(-2\sin x \cos x) + 16}\,dx$$

$$= \int_0^{\pi/4} \frac{\sin x + \cos x}{-9(\sin^2 x + \cos^2 x - 2\sin x \cos x - 1) + 16}\,dx$$

$[\because \sin^2 x + \cos^2 x = 1]$ (1)

$$= \int_0^{\pi/4} \frac{\sin x + \cos x}{25 - 9(\sin x - \cos x)^2}\,dx \quad (1)$$

$[\because a^2 + b^2 - 2ab = (a - b)^2]$

Put $\sin x - \cos x = t \Rightarrow (\cos x + \sin x)dx = dt$

Also, when, $x = 0$, then $t = -1$

and when, $x = \frac{\pi}{4}$, then $t = 0$

$$\therefore \quad I = \int_{-1}^{0} \frac{dt}{25 - 9t^2} = \frac{1}{9}\int_{-1}^{0} \frac{dt}{\left(\frac{5}{3}\right)^2 - t^2} \quad (1)$$

$$= \frac{1}{9} \times \frac{1}{2 \times \frac{5}{3}}\left[\log\left|\frac{\frac{5}{3} + t}{\frac{5}{3} - t}\right|\right]_{-1}^{0} \quad (1)$$

$$\left[\because \int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\left|\frac{a + x}{a - x}\right| + C\right]$$

$$= \frac{1}{30}\left[\log\left|\frac{5+3x}{5-3x}\right|\right]_{-1}^{0} = \frac{1}{30}\left[\log 1 - \log\left(\frac{2}{8}\right)\right]$$

$$= \frac{1}{30}\left[0 - \log\frac{1}{4}\right] \qquad [\because \log 1 = 0]$$

$$= \frac{1}{30}[-\log 4^{-1}] = \frac{1}{30}\log 4 \quad [\because \log m^n = n\log m] \text{ (1)}$$

53. Let $I = \int_1^3 (x^2 + 3x + e^x)\, dx$ ...(1)

On comparing the given integral with $\int_a^b f(x)\, dx$, we get

$$f(x) = x^2 + 3x + e^x,\ a = 1 \text{ and } b = 3$$

By definition,

$$\int_a^b f(x)dx = \lim_{h \to 0} h \begin{bmatrix} f(a) + f(a+h) + f(a+2h) \\ +\ldots + f(a+(n-1)h) \end{bmatrix} \quad \text{(1)}$$

Where $h = \frac{b-a}{n} \to 0$ as $n \to \infty$

$$\therefore \int_1^3 f(x)\, dx = \lim_{h \to 0} h \begin{bmatrix} f(1) + f(1+h) + f(1+2h) \\ +\ldots + f(1+(n-1)h) \end{bmatrix} \ldots\text{(i)}$$

Here, $nh = 3 - 1 = 2$

$$f(1) = 1^2 + 3(1) + e^1$$

$$f(1+h) = (1+h)^2 + 3(1+h) + e^{1+h}$$

$$f(1+(n-1)h) = (1+(n-1)h)^2 + 3(1+(n-1)h) + e^{1+(n-1)h}$$

$$\vdots \qquad \vdots \qquad \vdots \quad \text{(1)}$$

On substituting these values in Eq. (i), we get

$$\int_1^3 (x^2 + 3x + e^x)\, dx$$

$$= \lim_{h \to 0} h \begin{Bmatrix} [1^2 + (1+h)^2 + \ldots + (1+(n-1)h)^2] \\ +3[1 + (1+h) + \ldots + (1+(n-1)h)] \\ + [e^1 + e^{1+h} + \ldots + e^{1+(n-1)h}] \end{Bmatrix}$$

$$= \lim_{h \to 0} h \begin{Bmatrix} [1^2 + (1^2 + h^2 + 2h) + \ldots + (1^2 + (n-1)^2h^2 \\ +2(n-1)h] + 3[n + h(1 + 2 + \ldots + (n-1))] \\ + e[1 + e^h + \ldots + e^{(n-1)h}] \end{Bmatrix} \quad \text{(1)}$$

$$= \lim_{h \to 0} h \begin{Bmatrix} (1^1 + 1^2 + \ldots + 1^2) + h^2(1^2 + 2^2 + \ldots + (n-1)^2) \\ + 2h(1 + 2 + \ldots + (n-1)) + 3\left[n + \frac{h(n-1)n}{2}\right] \\ + e\left[1 \cdot \frac{e^{nh} - 1}{e^h - 1}\right] \end{Bmatrix}$$

$$\left[\begin{array}{l} \because 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} \\ \text{and } 1, e^h, \ldots, e^{(n-1)h} \text{ form a GP with } a = 1,\ r = e^h \end{array}\right]$$

$$= \lim_{h \to 0} h \begin{Bmatrix} \left[n + h^2 \frac{(n-1)n(2n-1)}{6} + \frac{2h(n-1)n}{2}\right] \\ + 3\left[n + \frac{n(n-1)h}{2}\right] + e\frac{(e^{nh} - 1)}{e^h - 1} \end{Bmatrix}$$

$$= \lim_{h \to 0} \begin{Bmatrix} nh + \frac{(nh-h)\, nh\, (2nh-h)}{6} + (nh-h)(nh) \\ + 3nh + 3\frac{(nh)(nh-h)}{2} + \frac{he(e^{nh} - 1)}{e^h - 1} \end{Bmatrix}$$

$$= \lim_{h \to 0} \begin{Bmatrix} 2 + \frac{(2-h)(2)(4-h)}{6} + (2-h)(2) + 6 \\ + 3\frac{2(2-h)}{2} + e\frac{(e^2 - 1)}{\left(\frac{e^h - 1}{h}\right)} \end{Bmatrix} \quad [\because nh = 2]$$

(1)

$$= \lim_{h \to 0} \begin{Bmatrix} 2 + \frac{1}{3}(2-h)(4-h) + 2(2-h) + 6 \\ + 3(2-h) + \frac{e(e^2 - 1)}{\frac{(e^h - 1)}{h}} \end{Bmatrix}$$

$$= 2 + \frac{1}{3} \times 2 \times 4 + 4 + 6 + 6 + \frac{e(e^2 - 1)}{\lim_{h \to 0}\left(\frac{e^h - 1}{h}\right)}$$

$$= \frac{8}{3} + 18 + e(e^2 - 1) \qquad \left[\because \lim_{x \to 0} \frac{e^x - 1}{x} = 1\right]$$

$$= \frac{54 + 8}{3} + e(e^2 - 1) = \frac{62}{3} + e(e^2 - 1)$$

(2)

54. Here, $a = 1$, $b = 3$, $f(x) = 3x^2 + 2x + 1$

Now, $nh = b - a = 3 - 1 = 2$ (1)

$$\int_1^3 (3x^2 + 2x + 1)\, dx$$

$$= \lim_{h \to 0} h[f(1) + f(1+h) + f(1+2h) + \ldots + f(1+(n-1)h)]$$

$$= \lim_{h \to 0} h[6 + \{3(1 + h^2 + 2h) + 2(1+h) + 1\}$$

$$+ \{3(1 + 4h^2 + 4h) + 2(1+2h) + 1\} + \ldots$$

$$\{3(1 + (n-1)^2h^2 + 2(n-1)h + 2(1 + (n-1)h) + 1\}] \text{ (1)}$$

$$= \lim_{h \to 0} h[6n + 8h(1 + 2 + \ldots (n-1)) + 3h^2(1^2 + 2^2$$

$$+ \ldots (n-1)^2] \text{ (1)}$$

$$= \lim_{h \to 0} 6hn + \frac{8(nh-h)(nh)}{2} + \frac{3(nh-h)(nh)(2hn-h)}{6}$$

$$= 6(2) + \frac{8(2)(2)}{2} + \frac{3(2-0)(2)(4)}{6}$$

$$= 12 + 16 + 8 = 36 \quad \text{(1)}$$

(1)

55. Let $I = \int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx$ ...(i)

Using $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, we get

$$I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right) \sin\left(\frac{\pi}{2} - x\right) \cos\left(\frac{\pi}{2} - x\right)}{\sin^4\left(\frac{\pi}{2} - x\right) + \cos^4\left(\frac{\pi}{2} - x\right)} dx$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right) \cos x \sin x}{\cos^4 x + \sin^4 x} dx \quad \text{...(ii)}$$

$\left[\because \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta \text{ and } \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta\right]$ (1)

On adding Eqs. (i) and (ii), we get

$$2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{\cos x \sin x}{\sin^4 x + \cos^4 x} dx$$

$$\Rightarrow I = \frac{\pi}{4} \int_0^{\pi/2} \frac{\sin x \cos x}{(\sin^2 x)^2 + (1 - \sin^2 x)^2} dx$$

Put $\sin^2 x = t$

$\Rightarrow \quad 2\sin x \cos x\, dx = dt$

$\Rightarrow \quad \sin x \cos x\, dx = \frac{dt}{2}$

Lower limit when $x = 0$, then

$t = \sin 0 = 0$

Upper limit when $x = \frac{\pi}{2}$, then

$t = \sin^2 \frac{\pi}{2} = 1$ (1)

$$\therefore \quad I = \frac{\pi}{4} \int_0^1 \frac{1}{t^2 + (1-t)^2} \frac{dt}{2}$$

$$\Rightarrow I = \frac{\pi}{8} \int_0^1 \frac{1}{t^2 + (1 + t^2 - 2t)} dt \quad (1)$$

$$\Rightarrow I = \frac{\pi}{8} \int_0^1 \frac{1}{2t^2 - 2t + 1} dt$$

$$\Rightarrow I = \frac{\pi}{16} \int_0^1 \frac{1}{t^2 - t + \frac{1}{2}} dt$$

$$\Rightarrow I = \frac{\pi}{16} \int_0^1 \frac{1}{t^2 - t + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + \frac{1}{2}} dt \quad (1)$$

$$\Rightarrow I = \frac{\pi}{16} \int_0^1 \frac{1}{\left(t - \frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} dt$$

$$\Rightarrow I = \frac{\pi}{16} \frac{1}{1/2} \left[\tan^{-1}\left(\frac{t - \frac{1}{2}}{\frac{1}{2}}\right)\right]_0^1$$

$\left[\because \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C\right]$ (1)

$$\Rightarrow I = \frac{\pi}{8}\left[\tan^{-1} 2\left(1 - \frac{1}{2}\right) - \tan^{-1} 2\left(0 - \frac{1}{2}\right)\right]$$

$$\Rightarrow I = \frac{\pi}{8}[\tan^{-1}(1) - \tan^{-1}(-1)]$$

$\left[\because \tan^{-1}(-1) = -\tan^{-1}(1) = -\frac{\pi}{4}\right]$

$$\Rightarrow I = \frac{\pi}{8}\left[\frac{\pi}{4} + \frac{\pi}{4}\right] = \frac{\pi^2}{16} \quad (1)$$

56. Let $I = \int_1^3 (e^{2-3x} + x^2 + 1)\,dx$

On comparing the given integral with

$\int_a^b f(x)\,dx$,we get

$a = 1, b = 3, f(x) = e^{2-3x} + x^2 + 1$

As we know that,

$$\int_a^b f(x)\,dx = \lim_{h \to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots + f\{a + (n-1)h\}] \quad \text{...(i)}$$

where, $nh = b - a$.

Here, $nh = 3 - 1 = 2$

$f(a) = f(1) = e^{2-3(1)} + (1)^2 + 1$

$f(a+h) = f(1+h)$
$= e^{2-3(1+h)} + (1+h)^2 + 1$ (1)

$f(a+2h) = f(1+2h)$
$= e^{2-3(1+2h)} + (1+2h)^2 + 1$

... ...

... ...

$f[a + (n-1)h] = f[1 + (n-1)h] = e^{2-3[1+(n-1)h]}$
$+ [1 + (n-1)h]^2 + 1$ (1)

On substituting these values in Eq. (i), we get

$$\int_a^b f(x)\,dx = \lim_{h \to 0} h[\{e^{2-3(1)} + e^{2-3(1+h)} + \ldots + e^{2-3[1+(n-1)h]}\} + \{1^2 + (1+h)^2 + (1+2h)^2 + \ldots + (1 + (n-1)h)^2\} + 1 + 1 + \ldots n \text{ terms}]$$

$$= \lim_{h \to 0} h[\{e^{-1} + e^{-1} \cdot e^{-3h} + e^{-1} \cdot e^{-3(2h)} + \ldots + e^{-1} \cdot e^{-3(n-1)h}\} + 1^2 + (1^2 + h^2 + 2h) + \{1^2 + (2h)^2 + 2(2h)\} + \ldots + \{1^2 + ((n-1)h)^2 + 2(n-1)h\} + n]$$

$$= \lim_{h \to 0} h\,[e^{-1}(1 + e^{-3h} + e^{2(-3h)} + \ldots + e^{(n-1)(-3h)})$$

$$+ \{n + h^2(1^2 + 2^2 + \ldots + (n-1)^2)$$

$$+ 2h(1 + 2 + \ldots + (n-1))\} + n] \quad (1)$$

$$= \lim_{h \to 0} h \left[e^{-1} \frac{1 - e^{(-3h)n}}{1 - e^{-3h}} + \left\{ n + h^2 \frac{(n-1)n(2n-1)}{6} + \frac{2h(n-1)n}{2} \right\} + n \right]$$

$$= \lim_{h \to 0} h \left[e^{-1} \frac{(1 - e^{-3nh})}{1 - e^{-3h}} + \left\{ n + h^2 \frac{(n-1)n(2n-1)}{6} + h(n-1)n \right\} + n \right] \quad (1)$$

$$= \lim_{h \to 0} \left\{ \frac{e^{-1}(1 - e^{-3nh})}{\frac{1 - e^{-3h}}{h}} \right\}$$

$$+ \lim_{h \to 0} \left[nh + \frac{(nh - h)(nh)(2nh - h)}{6} + (nh - h)nh \right]$$

$$+ \lim_{h \to 0} nh \quad (1)$$

$$= \left\{ \lim_{h \to 0} \frac{e^{-1}(1 - e^{-6})}{\left(\frac{1 - e^{-3h}}{h} \right)} + \lim_{h \to 0} \left[2 + \frac{(2-h)(2)(4-h)}{6} + (2-h)2 + \lim_{h \to 0} 2 \right] \right\}$$

$$[\because nh = 2]$$

$$= e^{-1}(1 - e^{-6}) \times \frac{1}{\lim_{h \to 0} \left(\frac{1 - e^{-3h}}{h} \right)}$$

$$+ \left[2 + \frac{2 \times 2 \times 4}{6} + 4 \right] + 2$$

$$= (e^{-1} - e^{-7}) \left[\frac{1}{-3 \lim_{h \to 0} \left(\frac{1 - e^{-3h}}{-3h} \right)} \right] + \frac{26}{3} + 2$$

$$= \left(\frac{e^{-1} - e^{-7}}{-3(-1)} \right) + \frac{32}{3} \qquad \left[\because \lim_{h \to 0} \frac{1 - e^h}{h} = -1 \right]$$

$$= \frac{-e^{-1}(e^{-6} - 1)}{3} + \frac{32}{3} \quad (1)$$

57. First, convert the denominator in the form of $(\cos x - \sin x)$, then put $\cos x - \sin x = t$ and simplify it.

Let $I = \int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx$

$$\Rightarrow \quad I = \int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16(1 + \sin 2x - 1)} dx \quad (1)$$

[adding and subtracting 16 from denominator]

$$\Rightarrow \quad I = \int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16[1 - (1 - \sin 2x)]} dx$$

$$\Rightarrow \quad I = \int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16\left[1 - (\cos^2 x + \sin^2 x - 2 \sin x \cos x)\right]} dx$$

$$\left[\because 1 = \cos^2 x + \sin^2 x \text{ and } \sin 2x = 2 \sin x \cos x \right]$$

$$\Rightarrow \quad I = \int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16[1 - (\cos x - \sin x)^2]} dx \quad (1)$$

Put $\cos x - \sin x = t$

$$\Rightarrow \quad (-\sin x - \cos x)\, dx = dt$$

$$\Rightarrow \quad (\sin x + \cos x)\, dx = -\, dt$$

Lower limit when $x = 0$, then $t = \cos 0 - \sin 0 = 1$

Upper limit when $x = \frac{\pi}{4}$, then

$$t = \cos \frac{\pi}{4} - \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0. \quad (1)$$

$$\therefore \quad I = \int_1^0 \frac{-dt}{9 + 16(1 - t^2)}$$

$$\Rightarrow \quad I = \int_0^1 \frac{dt}{9 + 16(1 - t^2)}$$

$$= \int_0^1 \frac{dt}{25 - 16t^2}$$

$$= \frac{1}{16} \int_0^1 \frac{dt}{\left(\frac{5}{4}\right)^2 - t^2}$$

$$= \frac{1}{2 \times \frac{5}{4} \times 16} \left[\log \left| \frac{5 + 4t}{5 - 4t} \right| \right]_0^1$$

$$\left[\because \int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \left| \frac{a + x}{a - x} \right| + C \right]$$

$$= \frac{1}{40} \left[\log \left| \frac{5 + 4}{5 - 4} \right| - \log \left| \frac{5}{5} \right| \right]$$

$$= \frac{1}{40}\left[\log\left(\frac{9}{1}\right) - \log\left(\frac{5}{5}\right)\right]$$

$$= \frac{1}{40}(\log 9 - \log 1) = \frac{1}{40}(\log 9) \quad \textbf{(2)}$$

$[\because \log 1 = 0]$

$$\Rightarrow I = \frac{1}{40}\log(3)^2$$

$$= \frac{2}{40}\log 3 \quad [\because \log a^n = n\log a]$$

$$\therefore \quad I = \frac{1}{20}\log 3 \quad \textbf{(1)}$$

58. Let $I = \int_0^\pi \frac{x}{a^2\cos^2 x + b^2\sin^2 x}dx \quad \ldots(i)$

$$\Rightarrow \quad I = \int_0^\pi \frac{(\pi - x)}{a^2\cos^2(\pi - x) + b^2\sin^2(\pi - x)}dx$$

$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$\Rightarrow \quad I = \int_0^\pi \frac{(\pi - x)}{a^2\cos^2 x + b^2\sin^2 x}dx \quad \ldots(ii)\ \textbf{(1)}$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^\pi \frac{(x + \pi - x)}{a^2\cos^2 x + b^2\sin^2 x}dx$$

$$\Rightarrow \quad 2I = \pi\int_0^\pi \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$$

Now, we know that,

$$\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx, \text{ if } f(2a - x) = f(x)$$

Here, $a^2\cos^2(\pi - x) + b^2\sin^2(\pi - x)$

$$= a^2\cos^2 x + b^2\sin^2 x \quad \textbf{(1)}$$

$$\therefore \quad 2I = 2\pi\int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$$

On dividing numerator and denominator by $\cos^2 x$, we get

$$2I = 2\pi\int_0^{\pi/2} \frac{\sec^2 x}{a^2 + b^2\tan^2 x}dx \quad \textbf{(1)}$$

Put $\quad \tan x = t \Rightarrow \sec^2 x\,dx = dt$

Lower limit when $x = 0$, then $t = \tan 0 = 0$

Upper limit when $x = \frac{\pi}{2}$, then $t = \tan\frac{\pi}{2} = \infty$. **(1)**

$$\therefore \quad I = \pi\int_0^\infty \frac{dt}{a^2 + b^2t^2}$$

$$= \pi\int_0^\infty \frac{dt}{a^2 + (bt)^2} = \frac{\pi}{b^2}\int_0^\infty \frac{dt}{\left(\frac{a}{b}\right)^2 + t^2}$$

$$\Rightarrow I = \frac{\pi}{ab}\left[\tan^{-1}\frac{bt}{a}\right]_0^\infty \left[\because \int\frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C\right]$$

$$\Rightarrow I = \frac{\pi}{ab}[\tan^{-1}\infty - \tan^{-1}0] \quad \textbf{(1)}$$

$$\Rightarrow I = \frac{\pi}{ab}\left[\frac{\pi}{2} - 0\right]$$

$$\left[\begin{array}{l}\because \tan^{-1}\infty = \tan^{-1}\left(\tan\frac{\pi}{2}\right) = \frac{\pi}{2} \\ \text{and } \tan^{-1}0 = \tan^{-1}(\tan 0^\circ) = 0\end{array}\right]$$

$$\therefore \quad I = \frac{\pi^2}{2ab} \quad \textbf{(1)}$$

59. Let $I = \int_0^\pi \frac{x\tan x}{\sec x + \tan x}dx \quad \ldots(i)$

$$\Rightarrow I = \int_0^\pi \frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)}dx$$

$$\left[\because \int_0^a f(x) = \int_0^a f(a - x)dx\right]$$

$$\Rightarrow I = \int_0^\pi \frac{-(\pi - x)\tan x}{-\sec x - \tan x}dx$$

$$\Rightarrow I = \int_0^\pi \frac{(\pi - x)\tan x}{\sec x + \tan x}dx \quad \ldots(ii)\ \textbf{(1)}$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^\pi \frac{\pi\tan x}{\sec x + \tan x}dx$$

$$= \pi\int_0^{\pi/2} \frac{\tan x}{\sec x + \tan x}dx + \pi\int_{\pi/2}^\pi \frac{\tan x}{\sec x + \tan x}dx$$

$$= \pi\int_0^{\pi/2} \frac{\sin x}{1 + \sin x}dx + \pi\int_{\pi/2}^\pi \frac{\sin x}{1 + \sin x}dx \quad \textbf{(1)}$$

$$= \pi\int_0^\pi \frac{\sin x}{1 + \sin x}dx = \pi\int_0^\pi \frac{1 + \sin x - 1}{1 + \sin x}dx \quad \textbf{(1)}$$

[adding and subtracting 1 in numerator]

$$= \pi\int_0^\pi \left(\frac{1 + \sin x}{1 + \sin x} - \frac{1}{1 + \sin x}\right)dx$$

$$= \pi\int_0^\pi \left(1 - \frac{1}{1 + \sin x}\right)dx$$

$$= \pi\left[\int_0^\pi 1\,dx - \int_0^\pi \frac{1}{1 + \sin x}dx\right]$$

$$= \pi\left[\int_0^\pi 1\,dx - 2\int_0^{\pi/2} \frac{1}{1 + \sin x}dx\right] \quad \textbf{(1½)}$$

$$\left[\begin{array}{l}\because \int_0^{2a} f(x) = 2\int_0^a f(x)dx, \\ \quad\text{if } f(2a - x) = f(x)\end{array}\right]$$

$$= \pi\left[\int_0^{\pi} dx - 2\int_0^{\pi/2} \frac{1}{1+\sin x} \times \frac{1-\sin x}{1-\sin x}\, dx\right]$$

$$= \pi\left[[x]_0^{\pi} - 2\int_0^{\pi/2} \frac{1-\sin x}{\cos^2 x}\, dx\right]$$

$[\because \sin^2 x + \cos^2 x = 1 \Rightarrow 1 - \sin^2 x = \cos^2 x]$

$$= \pi\left[[x]_0^{\pi} - 2\int_0^{\pi/2}\left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}\right) dx\right]$$

$$= \pi\left[(\pi - 0) - 2\int_0^{\pi/2} (\sec^2 x - \sec x \tan x)\, dx\right]$$

$$= \pi\{\pi - 2[\tan x - \sec x]_0^{\pi/2}\}$$ **(1½)**

$$= \pi\left\{\pi - \lim_{x \to \frac{\pi}{2}} 2(\tan x - \sec x) + 2(\tan 0 - \sec 0)\right\}$$

$$\Rightarrow 2I = \pi\{\pi - 0 + 2(0-1)\} = \pi(\pi - 2)$$

$$\therefore \quad I = \frac{\pi}{2}(\pi - 2)$$ **(1)**

60. Let $I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\cot x}}$

$$\Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx \quad \text{...(i)}$$ **(1)**

We know that,

$$\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$$ **(1)**

On applying this property in Eq. (i), we get

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin\left(\frac{\pi}{6} + \frac{\pi}{3} - x\right)}}{\sqrt{\cos\left(\frac{\pi}{6} + \frac{\pi}{3} - x\right)} + \sqrt{\sin\left(\frac{\pi}{6} + \frac{\pi}{3} - x\right)}}\, dx$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\cos\left(\frac{\pi}{2} - x\right)} + \sqrt{\sin\left(\frac{\pi}{2} - x\right)}}\, dx$$

$$\Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx \quad \text{...(ii)}$$ **(1)**

$\left[\because \cos\left(\frac{\pi}{2} - x\right) = \sin x \text{ and } \sin\left(\frac{\pi}{2} - x\right) = \cos x\right]$ **(1)**

On adding Eqs. (i) and (ii), we get

$$2I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx$$ **(1)**

$$\Rightarrow \quad 2I = \int_{\pi/6}^{\pi/3} (1)\, dx$$

$$\Rightarrow \quad 2I = [x]_{\pi/6}^{\pi/3}$$

$$\Rightarrow \quad 2I = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

$$\therefore \quad I = \frac{\pi}{12}$$ **(1)**

61. In limit of a sum, use the relation

$$\int_a^b f(x)dx = \lim_{h \to 0} h[f(a) + f(a+h) + f(a+2h) + \dots + f\{a + (n-1)h\}]$$

Further, solve it and get the desired result, where $h = (b-a)/n$.

We have, $\int_1^3 (3x^2 + 1)\, dx$

Here, $a = 1$, $b = 3$, $nh = b - a = 3 - 1 = 2$

and $\quad f(x) = 3x^2 + 1$

$$\Rightarrow \quad f(1) = 3(1)^2 + 1 = 4$$ **(1)**

$$f(1+h) = 3(1+h)^2 + 1 = 4 + 6h(1) + 3h^2(1)^2$$

$$f(1+2h) = 3(1+2h)^2 + 1 = 4 + 6h(2) + 3h^2(2)^2$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$f[1 + (n-1)h] = 3[1 + (n-1)h]^2 + 1$$
$$= 4 + 6h(n-1) + 3h^2(n-1)^2$$ **(1)**

Now,

$$\int_1^3 (3x^2+1)dx = \lim_{h \to 0} h[f(1) + f(1+h) + f(1+2h) + \dots + \dots f(1 + (n-1)h)]$$ [by definition] **(1)**

$$= \lim_{h \to 0} h[4 + 4 + 6h(1) + 3h^2(1)^2 + 4 + 6h(2) + 3h^2(2)^2 + \dots + 4 + 6h(n-1) + 3h^2(n-1)^2]$$

$$= \lim_{h \to 0} h\left[\underbrace{4 + 4 + \dots + 4}_{n \text{ times}} + 6h\{1 + 2 + 3 + \dots + (n-1)\} + 3h^2\{1^2 + 2^2 + \dots + (n-1)^2\}\right]$$ **(1)**

$$= \lim_{h \to 0} h\left[4n + 6h \cdot \frac{n(n-1)}{2} + 3h^2 \frac{n(n-1)(2n-1)}{6}\right]$$

$$= \lim_{h \to 0}\left[4nh + \frac{6nh(nh-h)}{2} + \frac{3hn(nh-h)(2nh-h)}{6}\right]$$ **(1)**

$$= 4(2) + \frac{6(2)(2-0)}{2} + \frac{3 \times 2(2-0)(2 \times 2 - 0)}{6} \quad [\because nh = 2]$$

$$= 8 + 12 + 8 = 28$$ **(1)**

62. We know that, by limit of a sum, we have

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \dots + f\{a+(n-1)h\}]$$

where, $h = \frac{b-a}{n}$.

Let $I = \int_1^3 (2x^2 + 5x)\,dx$

Here, $a = 1, b = 3, f(x) = 2x^2 + 5x,$

and $nh = b - a = 3 - 1 = 2$

$\therefore \quad f(1) = 2(1)^2 + 5(1) = 2 + 5 = 7$

$f(1+h) = 2(1+h)^2 + 5(1+h)$

$= 2 + 2h^2 + 4h + 5 + 5h = 2h^2 + 9h + 7$

$f(1+2h) = 2(1+2h)^2 + 5(1+2h)$

$= 2 + 8h^2 + 8h + 5 + 10h$

$= 8h^2 + 18h + 7$

$\vdots \quad \vdots \quad \vdots$

$f\{1+(n-1)h\} = 2\{1+(n-1)h\}^2 + 5\{1+(n-1)h\}]$

$= 2 + 2(n-1)^2h^2 + 4(n-1)h + 5 + 5(n-1)h$

$= 2(n-1)^2h^2 + 9(n-1)h + 7$ **(1½)**

Now, $\int_1^3 (2x^2 + 5x)\,dx$

$= \lim_{h\to 0} h[f(1) + f(1+h) + f(1+2h) + \dots + f\{1+(n-1)h\}]$ **(1)**

On putting all above values, we get

$$\int_1^3 (2x^2+5x)\,dx = \lim_{h\to 0} h[7 + (2h^2 + 9h + 7) + (8h^2 + 18h + 7) + \dots + 2(n-1)^2h^2 + 9(n-1)h + 7]$$

$$= \lim_{h\to 0} h[7 + 7 + \dots + 7 \ n \text{ times}] + \lim_{h\to 0} h[2h^2 + 8h^2 + \dots + 2(n-1)^2h^2] + \lim_{h\to 0} h[9h + 18h + \dots + 9(n-1)h]$$ **(1½)**

$$= \lim_{h\to 0} 7nh + \lim_{h\to 0} 2h^3[1^2 + 2^2 + \dots + (n-1)^2] + \lim_{h\to 0} 9h^2[1 + 2 + \dots + (n-1)]$$

$$= \lim_{h\to 0} 7(2) + \lim_{h\to 0} \frac{2h^3\cdot n(n-1)(2n-1)}{6} + \lim_{h\to 0}\frac{9h^2\cdot n(n-1)}{2}$$

$$\left[\because 1^2 + 2^2 + \dots + (n-1)^2 = \frac{n(n-1)(2n-1)}{6},\ 1 + 2 + \dots + (n-1) = \frac{n(n-1)}{2} \text{ and } nh = 2\right]$$

$$= 14 + \lim_{h\to 0}\frac{nh(nh-h)(2nh-h)}{3} + \lim_{h\to 0}\frac{9}{2}\cdot nh(nh-h)$$ **(1)**

$$= 14 + \frac{2(2-0)(4-0)}{3} + \frac{9}{2}\cdot 2(2-0) \qquad [\because nh = 2]$$

$$= 14 + \frac{16}{3} + 18 = \frac{42 + 16 + 54}{3} = \frac{112}{3}$$ **(1)**

63. Let $I = \int_0^{\pi/4} (\sqrt{\tan x} + \sqrt{\cot x})\,dx$

$$= \int_0^{\pi/4} \frac{\sqrt{\sin x}}{\sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\sin x}}\,dx$$

$$= \int_0^{\pi/4} \frac{(\sin x + \cos x)}{\sqrt{\sin x \cos x}}\,dx$$

$$= \sqrt{2}\int_0^{\pi/4} \frac{(\sin x + \cos x)}{\sqrt{2\sin x\cos x}}\,dx$$ **(1½)**

$$= \sqrt{2}\int_0^{\pi/4} \frac{(\sin x + \cos x)}{\sqrt{1 + 2\sin x\cos x - 1}}\,dx$$

$$= \sqrt{2}\int_0^{\pi/4} \frac{(\sin x + \cos x)}{\sqrt{1 - (1 - 2\sin x\cos x)}}\,dx$$

$$= \sqrt{2}\int_0^{\pi/4} \frac{(\sin x + \cos x)}{\sqrt{1 - (\sin x - \cos x)^2}}\,dx$$ **(1½)**

Now, put $\sin x - \cos x = t$

$\Rightarrow \quad (\cos x + \sin x)\,dx = dt$

Also, when $x = 0$, then $t = -1$

and when $x = \frac{\pi}{4}$, then $t = 0$ **(1)**

$$\therefore \quad I = \sqrt{2}\int_{-1}^{0} \frac{dt}{\sqrt{1-t^2}} = \sqrt{2}[\sin^{-1} t]_{-1}^{0}$$ **(1)**

$$= \sqrt{2}[\sin^{-1}(0) - \sin^{-1} -1]$$

$$= \sqrt{2}[\sin^{-1}(0) + \sin^{-1}(1)]$$

$$= \sqrt{2}\left[\frac{\pi}{2}\right]$$

(1) Hence proved.

64. Let $I = \int_{\pi/4}^{\pi/2} \underset{\text{II}}{\cos 2x}\, \underset{\text{I}}{\log(\sin x)}\,dx$

$$= \left[\log(\sin x)\frac{\sin 2x}{2} - \int \frac{1}{\sin x}\cos x\cdot\frac{\sin 2x}{2}\,dx\right]_{\pi/4}^{\pi/2}$$

[using integration by parts]

$$= \left[\frac{\log(\sin x)\cdot\sin 2x}{2}\right]_{\pi/4}^{\pi/2} - \int_{\pi/4}^{\pi/2}\frac{\cos x\cdot(2\sin x\cos x)}{2\sin x}\,dx$$ **(2)**

$$= \frac{1}{2}\left[\log\left(\sin\frac{\pi}{2}\right)\cdot\sin\pi - \log\left(\sin\frac{\pi}{4}\right)\cdot\sin\frac{\pi}{2}\right]$$
$$-\int_{\pi/4}^{\pi/2}\cos^2 x\,dx$$
$$= \frac{1}{2}\left[0 - \log\frac{1}{\sqrt{2}}\right] - \int_{\pi/4}^{\pi/2}\frac{1+\cos 2x}{2}dx$$
$$= \frac{1}{4}\log 2 - \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_{\pi/4}^{\pi/2} \quad \textbf{(2)}$$
$$= \frac{1}{4}\log 2 - \frac{1}{2}\left[\left(\frac{\pi}{2} + \frac{\sin\pi}{2}\right) - \left(\frac{\pi}{4} + \frac{\sin\pi/2}{2}\right)\right]$$
$$= \frac{1}{4}\log 2 - \frac{1}{2}\left(\frac{\pi}{2} - \frac{\pi}{4} - \frac{1}{2}\right)$$
$$= \frac{1}{4}\log 2 - \frac{1}{2}\left(\frac{\pi}{4} - \frac{1}{2}\right)$$
$$= \frac{1}{4}\log 2 - \frac{\pi}{8} + \frac{1}{4} \quad \textbf{(2)}$$

65. Let $I = \int_0^{\pi} \frac{x\tan x}{\sec x\cdot\operatorname{cosec} x}dx$ ...(i)

Using the property $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$,

we get

$$I = \int_0^{\pi}\frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x)\operatorname{cosec}(\pi - x)}dx$$
$$\Rightarrow \quad I = \int_0^{\pi}\frac{(\pi - x)(-\tan x)}{-\sec x\operatorname{cosec} x}dx$$
$$\left[\because \tan(\pi - x) = -\tan x, \sec(\pi - x) = -\sec x \text{ and } \operatorname{cosec}(\pi - x) = \operatorname{cosec} x\right]$$
$$\Rightarrow \quad I = \int_0^{\pi}\frac{(\pi - x)\tan x}{\sec x\operatorname{cosec} x}dx \quad \text{...(ii) } \textbf{(2)}$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi}\frac{\pi\tan x}{\sec x\operatorname{cosec} x}dx$$
$$\Rightarrow \quad I = \frac{\pi}{2}\int_0^{\pi}\frac{\sin^2 x(\cos x)}{(\cos x)}dx$$
$$= \frac{\pi}{2}\int_0^{\pi}\sin^2 x\,dx$$
$$= \frac{\pi}{2}\int_0^{\pi}\left(\frac{1-\cos 2x}{2}\right)dx$$
$$[\because \cos 2\theta = 1 - 2\sin^2\theta] \ \textbf{(2)}$$
$$= \frac{\pi}{4}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi}$$
$$\Rightarrow \quad I = \frac{\pi}{4}\left[\pi - \frac{\sin 2\pi}{2} - 0 + \frac{\sin(0)}{2}\right]$$
$$\Rightarrow \quad I = \frac{\pi}{4}[\pi - 0] \Rightarrow I = \frac{\pi^2}{4} \quad \textbf{(2)}$$

66. Let $I = \int_0^{\pi/2} 2\sin x\cos x\tan^{-1}(\sin x)\,dx$

Put $\sin x = t \Rightarrow \cos x\,dx = dt$

Lower limit when $x = 0$, then $t = \sin 0 = 0$

Upper limit when $x = \frac{\pi}{2}$, then $t = \sin\frac{\pi}{2} = 1$.

$$\therefore \quad I = 2\int_0^1 t\times\tan^{-1}t\,dt \quad \textbf{(1)}$$

Applying integration by parts, taking $\tan^{-1} t$ as 1st function and t as 2nd function, we get

$$I = 2\left[\frac{t^2}{2}\times\tan^{-1}t\right]_0^1 - 2\int_0^1\frac{1}{1+t^2}\times\frac{t^2}{2}dt$$
$$\left[\because \frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}\right] \quad \textbf{(1)}$$
$$\Rightarrow \quad I = 2\left[\frac{t^2}{2}\times\tan^{-1}t\right]_0^1 - \int_0^1\frac{t^2}{1+t^2}dt \quad \textbf{(1)}$$
$$\Rightarrow \quad I = 2\times\frac{1}{2}\times\tan^{-1}(1) - \int_0^1\frac{1+t^2-1}{1+t^2}dt$$
$$\Rightarrow \quad I = 1\times\frac{\pi}{4} - \int_0^1\left(\frac{1+t^2}{1+t^2} - \frac{1}{1+t^2}\right)dt \quad \textbf{(1)}$$
$$\Rightarrow \quad I = \frac{\pi}{4} - \int_0^1\left(1 - \frac{1}{1+t^2}\right)dt$$
$$\Rightarrow \quad I = \frac{\pi}{4} - [t - \tan^{-1}t]_0^1 \quad \textbf{(1)}$$
$$= \frac{\pi}{4} - 1 + \tan^{-1}1$$
$$= \frac{\pi}{4} - 1 + \frac{\pi}{4} = \frac{2\pi}{4} - 1 \quad \left[\because 1 = \tan\frac{\pi}{4}\right]$$
$$\therefore \quad I = \left(\frac{\pi}{2} - 1\right) \quad \textbf{(1)}$$

67. Do same as Q. No. 60. $\left[\textbf{Ans. } \frac{\pi}{12}\right]$

68. Given integral is $\int_1^4 (x^2 - x)\,dx$.

Here, $a = 1, b = 4, f(x) = x^2 - x$

and $nh = b - a = 4 - 1 = 3$ **(1)**

Now,
$$f(a) = f(1) = (1)^2 - (1) = 1 - 1 = 0$$
$$f(a+h) = f(1+h)$$
$$= (1+h)^2 - (1+h)$$
$$= 1 + h^2 + 2h - 1 - h$$
$$= h^2 + h$$

$$\Rightarrow \quad f(a+2h) = f(1+2h)$$
$$= (1+2h)^2 - (1+2h)$$
$$= 1 + 4h^2 + 4h - 1 - 2h$$
$$= 4h^2 + 2h \qquad \textbf{(1)}$$
$$\vdots \qquad \vdots$$
$$f[a+(n-1)h] = f[1+(n-1)h]$$
$$= [1+(n-1)h]^2 - [1+(n-1)h]$$
$$= 1 + (n-1)^2 h^2 + 2(n-1)h - 1 - (n-1)h$$
$$= (n-1)^2 h^2 + (n-1)h \qquad \textbf{(1)}$$

$$\therefore \int_1^4 (x^2 - x)\, dx$$
$$= \lim_{h\to 0} h\begin{bmatrix} 0 + (h^2+h) + (4h^2+2h) \\ + \dots + (n-1)^2 h^2 + (n-1)h \end{bmatrix}$$
$$\left[\because \int_a^b f(x)\,dx = \lim_{h\to 0} h\begin{bmatrix} f(a) + f(a+h) + f(a+2h) \\ + \dots + f\{a+(n-1)h\} \end{bmatrix}\right]$$
$$= \lim_{h\to 0} h\begin{bmatrix} h^2\{1+4+\dots+(n-1)^2\} \\ + h\{1+2+\dots+(n-1)\} \end{bmatrix}$$
$$= \lim_{h\to 0} h\begin{bmatrix} h^2\{1^2+2^2+\dots+(n-1)^2\} \\ + h\{1+2+\dots+(n-1)\} \end{bmatrix} \qquad \textbf{(1)}$$
$$= \lim_{h\to 0} h\left[h^2 \frac{n(n-1)(2n-1)}{6} + h\frac{n(n-1)}{2}\right]$$
$$\left[\begin{aligned} &\because 1^2+2^2+3^2+\dots+(n-1)^2 \\ &\qquad = \frac{n(n-1)(2n-1)}{6} \\ &\text{and } 1+2+3+\dots+(n-1) = \frac{n(n-1)}{2} \end{aligned}\right]$$
$$= \lim_{h\to 0}\left[\frac{nh(nh-h)(2nh-h)}{6} + \frac{nh(nh-h)}{2}\right]$$
$$= \lim_{h\to 0}\left[\frac{3(3-h)(6-h)}{6} + \frac{3(3-h)}{2}\right] \qquad \textbf{(1)}$$
$$[\because nh = 3]$$
$$= \frac{3\cdot 3\cdot 6}{6} + \frac{3\cdot 3}{2} = 9 + \frac{9}{2} = \frac{27}{2} \qquad \textbf{(1)}$$

69. Do same as Q. No. 61. **[Ans. 4]**

70. Do same as Q. No. 68.

Hint Here, $a = 0, b = 2, nh = 2$ $\left[\textbf{Ans. } \frac{2}{3}\right]$

71. Do same as Q.No. 61. $\left[\textbf{Ans. } \frac{70}{3}\right]$

72. Do same as Q. No. 62. $\left[\textbf{Ans. } \frac{59}{6}\right]$

73. Do same as Q. No. 62. **[Ans. 34]**

Objective Questions

(For Complete Chapter)

1 Mark Questions

1. $\int \sqrt{1+\cos x}\, dx$ is equal to

(a) $2\sin\left(\frac{x}{2}\right) + C$ (b) $\sqrt{2}\sin\left(\frac{x}{2}\right) + C$

(c) $2\sqrt{2}\sin\left(\frac{x}{2}\right) + C$ (d) $\frac{1}{2}\sin\left(\frac{x}{2}\right) + C$

2. $\int \frac{1+x+\sqrt{x+x^2}}{\sqrt{x}+\sqrt{1+x}}\, dx$ is equal to

(a) $\frac{1}{2}\sqrt{1+x} + C$ (b) $\frac{2}{3}(1+x)^{3/2} + C$

(c) $\sqrt{1+x} + C$ (d) $2(1+x)^{3/2} + C$

3. $\int \frac{\sin 2x}{\sin^2 x + 2\cos^2 x}\, dx$ is equal to

(a) $-\log(1+\sin^2 x) + C$

(b) $\log(1+\cos^2 x) + C$

(c) $-\log(1+\cos^2 x) + C$

(d) $\log(1+\tan^2 x) + C$

4. $\int \frac{\sec^2(\sin^{-1} x)}{\sqrt{1-x^2}}\, dx$ is equal to

(a) $\sin(\tan^{-1} x) + C$ (b) $\tan(\sec^{-1} x) + C$

(c) $\tan(\sin^{-1} x) + C$ (d) $-\tan(\cos^{-1} x) + C$

5. If $\int \frac{x+2}{2x^2+6x+5}\, dx$

$$= P\int \frac{4x+6}{2x^2+6x+5}\, dx + \frac{1}{2}\int \frac{dx}{2x^2+6x+5},$$

then the value of P is

(a) $\frac{1}{3}$ (b) $\frac{1}{2}$

(c) $\frac{1}{4}$ (d) 2

6. $\int \frac{x^{e-1} + e^{x-1}}{x^e + e^x}\, dx$ is equal to

(a) $\log(x^e + e^x) + C$ (b) $e\log(x^e + e^x) + C$

(c) $\frac{1}{e}\log(x^e + e^x) + C$ (d) None of these

7. $\int \frac{dx}{x(x^7+1)}$ is equal to

(a) $\log\left(\frac{x^7}{x^7+1}\right)+C$ (b) $\frac{1}{7}\log\left(\frac{x^7}{x^7+1}\right)+C$

(c) $\log\left(\frac{x^7+1}{x^7}\right)+C$ (d) $\frac{1}{7}\log\left(\frac{x^7+1}{x^7}\right)+C$

8. $\int_0^{\pi/2}\left|\cos\left(\frac{x}{2}\right)\right| dx$ is equal to

(a) 1 (b) -2 (c) $\sqrt{2}$ (d) 0

9. $3a\int_0^1\left(\frac{ax-1}{a-1}\right)^2 dx$ is equal to

(a) $a-1+(a-1)^{-2}$ (b) $a+a^{-2}$

(c) $a-a^2$ (d) $a^2+\frac{1}{a^2}$

10. The value of $\int_0^1 \frac{dx}{e^x+e}$ is

(a) $\frac{1}{e}\log\left(\frac{1+e}{2}\right)$ (b) $\log\left(\frac{1+e}{2}\right)$

(c) $\frac{1}{e}\log(1+e)$ (d) $\log\left(\frac{2}{1+e}\right)$

11. The value of $\int_{-2}^{2}(x\cos x+\sin x+1)\,dx$ is

(a) 2 (b) 0 (c) -2 (d) 4

12. $\int_0^{\pi/2}\frac{\sin x-\cos x}{1+\sin x\cos x}dx$ is equal to

(a) 0 (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) π

13. $\int_0^{\pi}\sqrt{\frac{1+\cos 2x}{2}}\,dx$ is equal to

(a) 0 (b) 2 (c) 4 (d) -2

14. If $\int_0^a f(2a-x)\,dx=m$ and $\int_0^a f(x)\,dx=n$, then $\int_0^{2a} f(x)\,dx$ is equal to

(a) $2m+n$ (b) $m+2n$

(c) $m-n$ (d) $m+n$

Solutions

1. (c) $\int\sqrt{1+\cos x}\,dx=\int\sqrt{2\cos^2(x/2)}\,dx$

$=\sqrt{2}\int\cos(x/2)\,dx$

$=\sqrt{2}\,\frac{\sin(x/2)}{1/2}+C=2\sqrt{2}\sin(x/2)+C$

2. (b) $\int\frac{1+x+\sqrt{x+x^2}}{\sqrt{x}+\sqrt{1+x}}dx$

$=\int\frac{\sqrt{(1+x)^2}+\sqrt{x}\sqrt{x+1}}{\sqrt{x}+\sqrt{1+x}}dx$

$=\int\frac{\sqrt{1+x}(\sqrt{1+x}+\sqrt{x})}{(\sqrt{1+x}+\sqrt{x})}dx$

$=\int\sqrt{1+x}\,dx=\frac{2}{3}(1+x)^{3/2}+C$

3. (c) Let $I=\int\frac{\sin 2x}{\sin^2 x+2\cos^2 x}dx$

$=\int\frac{\sin 2x}{1-\cos^2 x+2\cos^2 x}dx$

$=\int\frac{\sin 2x}{1+\cos^2 x}dx$

On putting $1+\cos^2 x=t \Rightarrow -2\cos x\sin x\,dx=dt$

$\Rightarrow \quad -\sin 2x\,dx=dt$

$\therefore I=-\int\frac{dt}{t}=-\log|t|+C$

$=-\log|1+\cos^2 x|+C$

4. (c) Let $I=\int\frac{\sec^2(\sin^{-1}x)}{\sqrt{1-x^2}}dx$

On putting $\sin^{-1}x=t \Rightarrow \frac{dt}{dx}=\frac{1}{\sqrt{1-x^2}}$

$\Rightarrow \quad dt=\frac{1}{\sqrt{1-x^2}}dx$

$\therefore I=\int\sec^2 t\,dt$

$=\tan t+C=\tan(\sin^{-1}x)+C$

5. (c) $\int\frac{x+2}{2x^2+6x+5}dx=\frac{1}{4}\int\frac{4x+6+2}{2x^2+6x+5}dx$

$=\frac{1}{4}\int\frac{4x+6}{2x^2+6x+5}dx+\frac{1}{2}\int\frac{dx}{2x^2+6x+5}$

$\therefore \quad P=\frac{1}{4}$

6. (c) Let $I=\int\frac{x^{e-1}+e^{x-1}}{x^e+e^x}dx$

On putting $x^e+e^x=t \Rightarrow e\,(x^{e-1}+e^{x-1})\,dx=dt$

$\therefore \quad I=\frac{1}{e}\int\frac{dt}{t}=\frac{1}{e}\log t+C=\frac{1}{e}\log(x^e+e^x)+C$

7. (b) Let $I=\int\frac{dx}{x\,(x^7+1)}$

On putting $x^7 = t \Rightarrow dx = \dfrac{1}{7x^6}dt$

$$\therefore \quad I = \frac{1}{7}\int \frac{dt}{t(t+1)} = \frac{1}{7}\int\left(\frac{1}{t} - \frac{1}{t+1}\right)dt$$

$$= \frac{1}{7}[\log t - \log(t+1)] + C$$

$$= \frac{1}{7}\log\left(\frac{t}{t+1}\right) + C$$

$$= \frac{1}{7}\log\left(\frac{x^7}{x^7+1}\right) + C$$

8. (c) $\displaystyle\int_0^{\pi/2}\left|\cos\left(\frac{x}{2}\right)\right|dx = \int_0^{\pi/2}\cos\left(\frac{x}{2}\right)dx$

$$\left[\because 0 < x < \frac{\pi}{2}, \left|\cos\left(\frac{x}{2}\right)\right| = \cos\left(\frac{x}{2}\right)\right]$$

$$= 2\left[\sin\left(\frac{x}{2}\right)\right]_0^{\pi/2} = 2\left[\sin\frac{\pi}{4} - \sin 0\right]$$

$$= 2\times\frac{1}{\sqrt{2}} = \sqrt{2}$$

9. (a) $\displaystyle 3a\int_0^1\left(\frac{ax-1}{a-1}\right)^2 dx = \frac{3a}{(a-1)^2}\left[\frac{(ax-1)^3}{3}\times\frac{1}{a}\right]_0^1$

$$= \frac{1}{(a-1)^2}[(a-1)^3 + 1] = (a-1) + (a-1)^{-2}$$

10. (a) Let $\displaystyle I = \int_0^1 \frac{dx}{e^x + e} = \int_0^1 \frac{dx}{e^x\left(1 + \frac{e}{e^x}\right)}$

Put $1 + \dfrac{e}{e^x} = t \Rightarrow 0 - \dfrac{e}{e^x}dx = dt$

$$\Rightarrow \frac{1}{e^x}dx = -\frac{1}{e}dt$$

$$\therefore \quad I = -\frac{1}{e}\int_{1+e}^{2}\frac{1}{t}dt = \frac{-1}{e}[\log t]_{1+e}^{2}$$

$$= \frac{-1}{e}[\log 2 - \log(1+e)]$$

$$= \frac{-1}{e}\log\left(\frac{2}{1+e}\right) = \frac{1}{e}\log\left(\frac{1+e}{2}\right)$$

11. (d) $\displaystyle\int_{-2}^{2}(x\cos x + \sin x + 1)\,dx$

$$= \int_{-2}^{2}x\cos x\,dx + \int_{-2}^{2}\sin x\,dx + \int_{-2}^{2}1\,dx$$

$$= [x\sin x - \int\sin x\,dx]_{-2}^{2} + \int_{-2}^{2}\sin x\,dx + [x]_{-2}^{2}$$

$$= 2\sin 2 - 2\sin 2 - \int_{-2}^{2}\sin x\,dx$$

$$+ \int_{-2}^{2}\sin x\,dx + (2+2) = 4$$

12. (a) Let $\displaystyle I = \int_0^{\pi/2}\frac{\sin x - \cos x}{1 + \sin x\cos x}dx$ ...(i)

$$\Rightarrow I = \int_0^{\pi/2}\frac{\sin\left(\frac{\pi}{2} - x\right) - \cos\left(\frac{\pi}{2} - x\right)}{1 + \sin\left(\frac{\pi}{2} - x\right)\cos\left(\frac{\pi}{2} - x\right)}dx$$

$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi/2}\frac{\cos x - \sin x}{1 + \cos x\sin x}dx \quad \text{...(ii)}$$

$$\left[\because \sin\left(\frac{\pi}{2} - x\right) = \cos x \text{ and } \cos\left(\frac{\pi}{2} - x\right) = \sin x\right]$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi/2}\frac{0}{1 + \sin x\cos x}dx = 0 \Rightarrow I = 0$$

13. (b) $\displaystyle\int_0^{\pi}\sqrt{\frac{1 + \cos 2x}{2}}dx$

$$= \int_0^{\pi}\sqrt{\frac{2\cos^2 x}{2}}dx = \int_0^{\pi}\sqrt{\cos^2 x}\,dx$$

$$= \int_0^{\pi}|\cos x|\,dx \qquad [\because \sqrt{x^2} = |x|]$$

$$= \int_0^{\pi/2}\cos x\,dx - \int_{\pi/2}^{\pi}\cos x\,dx$$

$$\left[\because \cos x > 0 \text{ for } 0 < x < \frac{\pi}{2} \text{ and } \cos x < 0 \text{ for } \frac{\pi}{2} < x < \pi\right]$$

$$= [\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{\pi}$$

$$= [1 - 0] - [0 - 1] = 1 + 1 = 2$$

14. (d) $\displaystyle\int_0^{2a}f(x)\,dx = \int_0^a\{f(2a-x) + f(x)\}\,dx$

$$= \int_0^a f(2a-x)\,dx + \int_0^a f(x)\,dx = m + n$$