6.APPLICATION OF DERIVATIVES

Single Correct Answer Type

1. The function x^x decreases in the interval
a) $(0, e)$ b) $(0, 1)$ c) $\left(0, \frac{1}{e}\right)$ d) None of these

2. $f(x) = \begin{cases} 4x - x^3 + \ln(a^2 - 3a + 3), 0 \le x < 3 \\ x - 18, \quad x \ge 3 \end{cases}$
Complete set of values of a such that $f(x)$ as a local minima at $x = 3$ is
a) $[-1, 2]$ b) $(-\infty, 1) \cup (2, \infty)$ c) $[1, 2]$ d) $(-\infty, -1) \cup (2, \infty)$

3. If $a^2x^4 + b^2y^4 = c^6$, then the maximum value of xy is
a) $\frac{c^2}{\sqrt{ab}}$ b) $\frac{c^3}{ab}$ c) $\frac{c^3}{\sqrt{2ab}}$ d) $\frac{c^3}{2ab}$

4. The function $f(x) = \sin^4 x + \cos^4 x$ increases if
a) $0 < x < \pi/8$ b) $\pi/4 < x < 3\pi/8$ c) $3\pi/8 < x < 5\pi/8$ d) $5\pi/8 < x < 3\pi/4$

5. If there is an error of $k\%$ in measuring the edge of a cube, then the percent error in estimating its volume is
a) k b) $3k$ c) $\frac{k}{3}$ d) None of these

6. In [0, 1] Lagranges Mean value theorem is NOT applicable to
a) $f(x) = \begin{cases} \frac{1}{2} - x, & x < \frac{1}{2} \\ \left(\frac{1}{2} - x\right)^2, & x \ge \frac{1}{2} \end{cases}$ b) $f(x) = \begin{cases} \frac{\sin x}{x}, & x \ne 0 \\ 1, & x = 0 \end{cases}$
c) $f(x) = x|x|$ d) $f(x) = |x|$

7. A rectangle is inscribed in an equilateral triangle of side length $2a$ units. The maximum area of this rectangle can be
a) $\sqrt{3}a^2$ b) $\frac{\sqrt{3}a^2}{4}$ c) a^2 d) $\frac{\sqrt{3}a^2}{2}$

8. If the function $f(x) = \frac{K \sin x + 2 \cos x}{\sin x + \cos x}$ is strictly increasing for all values of x, then
a) $K < 1$ b) $K > 1$ c) $K < 2$ d) $K > 2$

9. If $f(x) = x^3 + 4x^2 + \lambda x + 1$ is a monotonically decreasing function of x in the largest possible interval $(-2, -2/3)$, then
a) $\lambda = 4$
b) $\lambda = 2$
c) $\lambda = -1$
d) λ has no real value

10. If $a + b + c = 0$, then the quadratic equation $3ax^2 + 2bx + c = 0$ has
a) At least one root in [0, 1] b) One root in [2, 3] and the other in [−2, −1]
c) Imaginary roots d) None of these

11. Let the function $f(x)$ be defined as follows
$$f(x) = \begin{cases} x^3 + x^2 - 10x, -1 \le x < 0 \\ \cos x, \quad 0 \le x < \pi/2 \\ 1 + \sin x, \quad \pi/2 \le x \le \pi \end{cases}$$
Then $f(x)$ has
a) A local minimum at $x = \pi/2$
b) A global maximum at $x = \pi/2$
c) An absolute minimum at $x = -1$
d) An absolute maximum at $x = \pi$

12. Let the function $g: (-\infty, \infty) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ be given by $g(u) = 2\tan^{-1}(e^u) - \frac{\pi}{2}$. Then, g is

a) Even and is strictly increasing in $(0, \infty)$
b) Odd and is strictly decreasing in $(-\infty, \infty)$
c) Odd and is strictly increasing in $(-\infty, \infty)$
d) Neither even nor odd, but is strictly increasing in $(-\infty, \infty)$

13. The length of the largest continuous interval in which the function $f(x) = 4x - \tan 2x$ is monotonic is

a) $\pi/2$
b) $\pi/4$
c) $\pi/8$
d) $\pi/16$

14. The distance between the origin and the tangent to the curve $y = e^{2x} + x^2$ drawn at the point $x = 0$ is

a) $\frac{1}{\sqrt{5}}$
b) $\frac{2}{\sqrt{5}}$
c) $\frac{-1}{\sqrt{5}}$
d) $\frac{2}{\sqrt{3}}$

15. Which of the following statements is true for the function

$$f(x) = \begin{cases} \sqrt{x}, & x \geq 1 \\ x^3, & 0 \leq x \leq 1 \\ \frac{x^3}{3} - 4x, & x < 0 \end{cases}$$

a) It is monotonic increasing $\forall\, x \in R$
b) $f'(x)$ fails to exist for three distinct real values of x
c) $f'(x)$ changes its sign twice as x varies from $-\infty$ to ∞
d) The function attains its extreme values at x_1 and x_2, such that $x_1, x_2 > 0$

16. The normal to the curve $x = a(\cos\theta + \theta\sin\theta), y = a(\sin\theta - \theta\cos\theta)$ at any point θ is such that

a) It makes a constant angle with the x-axis
b) It passes through the origin
c) It is at a constant distance from the origin
d) None of these

17. If m is the slope of a tangent to the curve $e^y = 1 + x^2$, then

a) $|m| > 1$
b) $m > 1$
c) $m > -1$
d) $|m| \leq 1$

18. If $A > 0, B > 0$ and $A + B = \frac{\pi}{3}$, then the maximum value of $\tan A \tan B$ is

a) $\frac{1}{\sqrt{3}}$
b) $\frac{1}{3}$
c) 3
d) $\sqrt{3}$

19. At any point on the curve $2x^2y^2 - x^4 = c$, the mean proportional between the abscissa and the difference between the abscissa and the subnormal drawn to the curve at the same point is equal to

a) Ordinate
b) Radius vector
c) x-intercept of tangent
d) Sub-tangent

20. The tangent to the curve $y = e^{kx}$ at a point (0, 1) meets the x-axis at $(a, 0)$ where $a \in [-2, -1]$, then $k \in$

a) $[-1/2, 0]$
b) $[-1, -1/2]$
c) $[0, 1]$
d) $[1/2, 1]$

21. The volume of the greatest cylinder which can be inscribed in a cone of height 30 cm and semi-vertical angle 30° is

a) $4000\,\pi/3$ cubic cm
b) $400\,\pi/3$ cubic cm
c) $4000\,\pi/\sqrt{3}$ cubic cm
d) None of these

22. The angel made by the tangent of the curve $x = a(1 + \sin t\cos t)$; $y = a(1 + \sin t)^2$ with the x-axis at any point on it is

a) $\frac{1}{4}(\pi + 2t)$
b) $\frac{1 - \sin t}{\cos t}$
c) $\frac{1}{4}(2t - \pi)$
d) $\frac{1 + \sin t}{\cos 2t}$

23. The value of a for which the function $f(x) = a\sin x + (1/3)\sin 3x$ has an extremum at $x = \pi/3$ is

a) 1
b) −1
c) 0
d) 2

24. Let $f(x)$ be a function such that $f'(x) = \log_{1/3}[\log_3(\sin x + a)]$. If $f(x)$ is decreasing for all real values of x, then

a) $a \in (1, 4)$
b) $a \in (4, \infty)$
c) $a \in (2, 3)$
d) $a \in (2, \infty)$

25. If $f(x) = x^3 + 7x - 1$, then $f(x)$ has a zero between $x = 0$ and $x = 1$. The theorem that best describes this is

a) Mean value theorem
b) Maximum-minimum value theorem
c) Intermediate value theorem
d) None of these

26. The triangle formed by the tangent to the curve $f(x) = x^2 + bx - b$ at the point $(1, 1)$ and the co-ordinate axes lies in the first quadrant. If its area is 2, then the value of b is

a) -1 b) 3 c) -3 d) 1

27. If $f(x) = x^3 + bx^2 + cx + d$ and $0 < b^2 < c$, then in $(-\infty, \infty)$

a) $f(x)$ is strictly increasing function b) $f(x)$ has a local maxima
c) $f(x)$ is strictly decreasing function d) $f(x)$ is bounded

28. A cube of ice melts without changing its shape at the uniform rate of $4\text{cm}^3/\text{min}$. The rate of change of the surface area of the cube, in cm^2/min, when the volume of the cube is $125\ \text{cm}^3$, is

a) -4 b) $-16/5$ c) $-16/6$ d) $-8/15$

29. Tangents are drawn to $x^2 + y^2 = 16$ from the point $P(0, h)$. These tangents meet the x-axis at $A$and B. If the area of triangle PAB is minimum, then

a) $h = 12\sqrt{2}$ b) $h = 6\sqrt{2}$ c) $h = 8\sqrt{2}$ d) $h = 4\sqrt{2}$

30. The minimum value of $e^{(2x^2-2x+1)\sin^2 x}$ is

a) e b) $1/e$ c) 1 d) 0

31. If $f(x)$ and $g(x)$ are differentiable functions for $0 \le x \le 1$ such that $f(0) = 10, g(0) = 2, f(1) = 2, g(1) = 4$, then in the interval $(0, 1)$

a) $f'(x) = 0$ for all x b) $f'(x) + 4g'(x) = 0$ for at least one x
c) $f'(x) = 2g'(x)$ for at most one x d) None of these

32. The function $f(x) = \sum_{K=1}^{5}(x-K)^2$ assumes the minimum value of x given by

a) 5 b) $\frac{5}{2}$ c) 3 d) 2

33. Let $f(x) = \int e^x (x-1)(x-2)dx$. Then f decreases in the interval

a) $(-\infty, -2)$ b) $(-2, -1)$ c) $(1, 2)$ d) $(2, +\infty)$

34. A value of C for which the conclusion of mean value theorem holds for the function $f(x) = \log_e x$ on the interval $[1, 3]$ is

a) $\frac{1}{2}\log_e 3$ b) $\log_3 e$ c) $\log_e 3$ d) $2\log_3 e$

35. On which of the following intervals is the function $x^{100} + \sin x - 1$ decreasing?

a) $(0, \pi/2)$ b) $(0, 1)$ c) $(\pi/2, \pi)$ d) None of these

36. Let C be the curve $y = x^3$ (where x takes all real values). The tangent at A meets the curve again at B. If the gradient at B is K times the gradient at A, then K is equal to

a) 4 b) 2 c) -2 d) $\frac{1}{4}$

37. Let $g(x) = 2f\left(\frac{x}{2}\right) + f(2-x)$and $f''(x) < 0\ \forall\ x \in (0, 2)$. Then $g(x)$ increases in

a) $(1/2, 2)$ b) $(4/3, 2)$ c) $(0, 2)$ d) $(0, 4/3)$

38. The curve given by $x + y = e^{xy}$ has a tangent parallel to the y-axis at the point

a) $(0, 1)$ b) $(1, 0)$ c) $(1, 1)$ d) None of these

39. The co-ordinates of a point on the parabola $y^2 = 8x$ whose distance from the circle $x^2 + (y+6)^2 = 1$ is minimum is

a) $(2, 4)$ b) $(2, -4)$ c) $(18, -12)$ d) $(8, 8)$

40. Consider the following statements is S and RS: Both $\sin x$ and $\cos x$ are decreasing functions in the interval $\left(\frac{\pi}{2}, \pi\right)$

R: If a differentiable function decreases in an interval (a, b), then its derivative also decreases in (a, b), which of the following is true?

a) Both S and R are wrong
b) Both S and R are correct, but R is not the correct explanation of S
c) S is correct and R is the correct explanation for S
d) S is correct and R is wrong

41. All possible values of x for which the function

$f(x) = x \text{ In } x - x + 1$ is possible is

a) $(1, \infty)$ b) $(1/e, \infty)$ c) $[e, \infty)$ d) $(0,1) \cup (1, \infty)$

42. If the length of sub-normal is equal to the length of sub-tangent at any point (3, 4) on the curve $y = f(x)$ and the tangent at (3, 4) to $y = f(x)$ meets the coordinate axes at A and B, then the maximum area of the triangle OAB, where O is origin, is

a) 45/2 b) 49/2 c) 25/2 d) 81/2

43. If at each point of the curve $y = x^3 = ax^2 + x + 1$, the tangent is inclined at an acute angle with the positive direction of the x-axis, then

a) $a > 0$ b) $a \leq \sqrt{3}$ c) $-\sqrt{3} \leq a \leq \sqrt{3}$ d) None of these

44. The value of c in Lagrange's theorem for the function $f(x) = \log \sin x$ in the interval $[\pi/6, 5\pi/6]$

a) $\pi/4$ b) $\pi/2$ c) $2\pi/3$ d) None of these

45. Let f be continuous and differentiable function such that $f(x)$ and $f'(x)$ have opposite signs everywhere. Then

a) f is increasing

b) f is decreasing

c) $|f|$ is non-monotonic

d) $|f|$ is decreasing

46. $f(x) = \begin{cases} 2 - |x^2 + 5x + 6|, & x \neq -2 \\ a^2 + 1, & x = -2 \end{cases}$, then the range of a, so that $f(x)$ has maxima at $x = -2$, is

a) $|a| \geq 1$ b) $|a| < 1$ c) $a > 1$ d) $a < 1$

47. In a $\Delta ABC, \angle B = 90°$and $b + a = 4$. The area of the triangle is maximum when $\angle C$ is

a) $\pi/4$ b) $\pi/6$ c) $\pi/3$ d) None of these

48. The curve represented parametrically by the equations $x = 2 \text{ In} \cot t + 1$ and $y = \tan t + \cot t$

a) Tangent and normal intersect at the point (2, 1)

b) Normal at $t = \pi/4$ is parallel to the y-axis

c) Tangent at $t = \pi/4$ is parallel to the line $y = x$

d) Tangent at $t = \pi/4$ is parallel to the x-axis

49. Which of the following statements is always true?

a) If $f(x)$ is increasing, then $f^{-1}(x)$ is decreasing

b) If $f(x)$ is increasing, then $\frac{1}{f(x)}$ is also increasing

c) If f and g are positive function and f is increasing and g is decreasing, then f/g is a decreasing function

d) If f and g are positive function and f is decreasing and g is increasing, then f/g is a decreasing function

50. Let $f: R \to R$ be a function such that $f(x) = ax + 3 \sin x + 4 \cos x$. Then $f(x)$ is invertible if

a) $a \in (-5, 5)$ b) $a \in (-\infty, 5)$ c) $a \in (-5, +\infty)$ d) None of these

51. The lines tangent to the curves $y^3 - x^2y + 5y - 2x = 0$ and $x^4 - x^3y^2 + 5x + 2y = 0$ at the origin intersect at an angle θ equal to

a) $\frac{\pi}{6}$ b) $\frac{\pi}{4}$ c) $\frac{\pi}{3}$ d) $\frac{\pi}{2}$

52. The number of solutions of the equation $x^3 + 2x^2 + 5x + 2 \cos x = 0$ in $[0, 2\pi]$ is

a) One b) Two c) Three d) Zero

53. If a variable tangent to the curve $x^2y = c^3$ makes intercepts a, b, on x-and y-axes, respectively, then the value of a^2b is

a) $27c^3$ b) $\frac{4}{27}c^3$ c) $\frac{27}{4}c^3$ d) $\frac{4}{9}c^3$

54. Let $f(x) = \begin{cases} |x|, \text{ for } 0 < |x| \leq 2 \\ 1, \quad \text{for } x = 0 \end{cases}$ then at $x = 0, f$ has

a) A local maximum b) No local maximum c) A local minimum d) No extremum

55. The largest term in the sequence $a_n = \frac{n^2}{n^3+200}$ is given by

a) $\frac{529}{49}$ b) $\frac{8}{89}$ c) $\frac{49}{543}$ d) None of these

56. The point on the curve $3y = 6x - 5x^3$, the normal at which passes through the origin, is
a) $(1, 1/3)$ b) $(1/3, 1)$ c) $(2, -28/3)$ d) None of these

57. If $f(x) = x + \sin x$; $g(x) = e^{-x}$; $u = \sqrt{c+1} - \sqrt{c}$; $v = \sqrt{c} - \sqrt{c-1}$; $(c > 1)$, then
a) $fog(u) < fog\ (v)$ b) $gof\ (u) < gof\ (v)$ c) $gof\ (u) > gof\ (v)$ d) $fog(u) < fog\ (v)$

58. If $f''(x) > 0, \forall\, x \in R, f'(3) = 0$ and $g(x) = f(\tan^2 x - 2\tan x + 4), 0 < x < \frac{\pi}{2}$, then $g(x)$ is increasing in
a) $\left(0, \frac{\pi}{4}\right)$ b) $\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$ c) $\left(0, \frac{\pi}{3}\right)$ d) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

59. If $f(x) = kx^3 - 9x^2 + 9x + 3$ is monotonoically increasing in R, then
a) $k < 3$ b) $k \leq 2$ c) $k \geq 3$ d) None of these

60. The greatest value of $f(x) = \cos(xe^{[x]} + 7x^2 - 3x)\,, x \in [-1, \infty)$ is (where $[\cdot]$ represents the greatest integer function)
a) -1 b) 1 c) 0 d) None of these

61. If $\phi(x)$ is a polynomial function and $\phi'(x) > \phi(x), \forall\, x \geq 1$ and $\phi(1) = 0$, then
a) $\phi(x) \geq 0, \forall\, x \geq 1$ b) $\phi(x) < 0, \forall\, x \geq 1$ c) $\phi(x) = 0, \forall\, x \geq 1$ d) None of these

62. A given right cone has a volume p, and the largest right circular cylinder that can be inscribed in the cone has a volume q. Then p: q is
a) 9:4 b) 8:3 c) 7:2 d) None of these

63. The slope of the tangent to the curve $y = \sqrt{4 - r^2}$ at the point, where the ordinate and the abscissa are equal, is
a) -1 b) 1 c) 0 d) None of these

64. If the function $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$, where $a > 0$, attains its maximum and minimum at p and q, respectively such that $p^2 = q$, then a equals to
a) 1 b) 2 c) $\frac{1}{2}$ d) 3

65. If a function $f(x)$ has $f'(a) = 0$ and $f''(a) = 0$, then
a) $x = a$ is a maximum for $f(x)$ b) $x = a$ is a minimum for $f(x)$
c) It is difficult to say a and b d) $f(x)$ is necessary a constant function

66. If $3\,(a + 2c) = 4\,(b + 3d)$, then the equation $ax^3 + bx^2 + cx + d = 0$ will have
a) No real solution b) At least one real root in $(-1, 0)$
c) At least one real root in (0, 1) d) None of these

67. The least value of a, for which the equation $\frac{4}{\sin x} + \frac{1}{1-\sin x} = a$ has at least one solution in the interval $(0, \pi/2)$, is
a) 9 b) 4 c) 8 d) 1

68. The radius of a right circular cylinder increases at the rate of 0.1 cm/min, and the height decreases at the rate of 0.2 cm/min. The rate of change of the volume of the cylinder, in cm^3/min, when the radius is 2 cm and the height is 3 cm is
a) $-2p$ b) $-\frac{8\pi}{5}$ c) $-\frac{3\pi}{5}$ d) $\frac{2\pi}{5}$

69. The function $f(x) = \frac{\text{In}\ (\pi+x)}{\text{In}\ (e+x)}$ is
a) Increasing in $(0, \infty)$ b) Decreasing in $(0, \infty)$
c) Increasing in $(0, \pi/e)$, decreasing in $(\pi/e, \infty)$ d) Decreasing in $(0, \pi/e)$, increasing in $(\pi/e, \infty)$

70. If $f(x) = 4x^3 - x^2 - 2x + 1$ and $g(x) = \begin{cases} \min\{f(t): 0 \leq t \leq x\}; & 0 \leq x \leq 1 \\ 3 - x; & 1 < x \leq 2 \end{cases}$ then $g\left(\frac{1}{4}\right) + g\left(\frac{3}{4}\right) + g\left(\frac{5}{4}\right)$ has the value equal to
a) 7/4 b) 9/4 c) 13/4 d) 5/2

71. Let $f'(x) = e^{x^2}$ and $f(0) = 10$. If $A < f(1) < B$ can be concluded from the mean value theorem, then the largest value of $(A - B)$ equals
a) e b) $1 - e$ c) $e - 1$ d) $1 + e$

72. If $f(1) = -2$ and $f'(x) \geq 4.2$ for $1 \leq x \leq 6$. The smallest possible value of $f(6)$ is
a) 9 b) 12 c) 15 d) 19

73. If the line joining the points (0, 3) and (5, −2) is a tangent to the curve $y = \frac{c}{x+1}$, then the value of c is

a) 1 b) −2 c) 4 d) None of these

74. If $f(x) = x^5 - 5x^4 + 5x^3 - 10$ has local maximum and minimum at $x = p$ and $x = q$, respectively, then $(p, q) =$

a) (0, 1) b) (1, 3) c) (1, 0) d) None of these

75. The maximum slope of the curve $y = -x^3 + 3x^2 + 9x - 27$ is

a) 0 b) 12 c) 16 d) 32

76. Let $f(x) = (1 + b^2)x^2 + 2bx + 1$ and let $m(b)$ be the minimum value of $f(x)$. As b varies, the range of $m(b)$ is

a) [0, 1] b) (0, 1/2] c) [1/2, 1] d) (0, 1]

77. If $x + 4y = 14$ is a normal to the curve $y^2 = \alpha x^3 - \beta$ at (2, 3), then the value of $\alpha + \beta$ is

a) 9 b) −5 c) 7 d) −7

78. The maximum value of $f(x) = \frac{x}{1+4x+x^2}$ is

a) $-\frac{1}{4}$ b) $-\frac{1}{3}$ c) $\frac{1}{6}$ d) $\frac{1}{5}$

79. The function $f(x) = \tan^{-1}(\sin x + \cos x)$ is an increasing function in

a) $\left(-\frac{\pi}{2}, \frac{\pi}{4}\right)$ b) $\left(0, \frac{\pi}{4}\right)$ c) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ d) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

80. Tangent of acute angle between the curves $y = |x^2 - 1|$ and $y = \sqrt{7 - x^2}$ at their points of intersection is

a) $\frac{5\sqrt{3}}{2}$ b) $\frac{3\sqrt{5}}{2}$ c) $\frac{5\sqrt{3}}{4}$ d) $\frac{3\sqrt{5}}{4}$

81. $f: R \to R, f(x)$ is differentiable such that $f(f(x)) = k(x^5 + x), (k \neq 0)$, then $f(x)$ is always

a) Increasing b) Decreasing

c) Either increasing or decreasing d) Non-monotonic

82. $f(x) = |x \log_e x|$ monotonically decreases in

a) $(0, 1/e)$ b) $(1/e, 1)$ c) $(1, \infty)$ d) $(1/e, \infty)$

83. Let $f(x) = \begin{cases} x + 2, -1 \le x < 0 \\ 1, \quad x = 0 \\ \frac{x}{2}, \quad 0 < x \le 1 \end{cases}$

Then on [−1, 1], this function has

a) A minimum b) A maximum

c) Either a maximum or a minimum d) Neither a maximum nor a minimum

84. The number of values of k for which the equation $x^3 - 3x + k = 0$ has two distinct roots lying in the interval (0, 1) is

a) Three b) Two c) Infinitely many d) Zero

85. A rectangle of the greatest area is inscribed in a trapezium $ABCD$. One of whose non-parallel sides AB is perpendicular to the base, so that one of the rectangle's side lies on the larger base of the treapezium. The base of trapezium are 6 and 10 cm and AB is 8 cm long. Then the maximum area of the rectangle is

a) 24 sq. cm b) 48 sq. cm c) 36 sq. cm d) None of these

86. Consider the function $f(x) = \begin{cases} x \sin\frac{\pi}{x}, \text{for } x > 0 \\ 0, \quad \text{for } x = 0 \end{cases}$ then the number of points in (0, 1) where the derivative $f'(x)$ vanishes is

a) 0 b) 1 c) 2 d) Infinite

87. The function $f(x) = x(x + 4)e^{-x/2}$ has its local maxima at $x = a$, then

a) $a = 2\sqrt{2}$ b) $a = 1 - \sqrt{3}$ c) $a = -1 + \sqrt{3}$ d) $a = -4$

88. $f(x) = (x - 8)^4(x - 9)^5, 0 \le x \le 10$, monotonically decreases in

a) $\left(\frac{76}{9}, 10\right]$ b) $\left(8, \frac{76}{9}\right)$ c) $[0, 8)$ d) $\left(\frac{76}{9}, 10\right]$

89. The x- intercept of the tangent at any arbitrary point of the curve $\frac{a}{x^2} + \frac{b}{y^2} = 1$ is proportional to

a) Square of the abscissa of the point of tangency
b) Square root of the abscissa of the point of tangency
c) Cube of the abscissa of the point of tangency
d) Cube root of the abscissa of the point of tangency

90. The vertices of a triangle are $(0, 0)$, $(x\cos x)$ and $(\sin^3 x, 0)$where $0 < x < \frac{\pi}{2}$. The maximum area for such a triangle in sq. units is
a) $\frac{3\sqrt{3}}{32}$ b) $\frac{\sqrt{3}}{32}$ c) $\frac{4}{32}$ d) $\frac{6\sqrt{3}}{32}$

91. Tangent is drawn to ellipse $\frac{x^2}{27} + y^2 = 1$ at $(3\sqrt{3}\cos\theta, \sin\theta)$ (where$\theta \in (0, \pi/2)$). Then the value of θ such that sum of intercepts on axes made by this tangent is minimum, is
a) $\pi/3$ b) $\pi/6$ c) $\pi/8$ d) $\pi/4$

92. If $f(x) = x^\alpha, \log x$ and $f(0) = 0$, then the value of α for which Rolle's theorem can be applied in $[0, 1]$ is
a) -2 b) -1 c) 0 d) 1/2

93. If the normal to the curve $y = f(x)$ at the point$(3, 4)$ makes an angle $\frac{3\pi}{4}$ with the positive x-axis, then $f'(3)$ is equal to
a) -1 b) $-\frac{3}{4}$ c) $\frac{4}{5}$ d) 1

94. The three sides of a trapezium are equal, each being 8 cm. The area of the trapezium, when it is maximum, is
a) $24\sqrt{3}$ sq. cm b) $48\sqrt{3}$ sq. cm c) $72\sqrt{3}$ sq. cm d) None of these

95. Let $f(x) = \cos\pi x + 10x + 3x^2 + x^3, -2 \le x \le 3$. The absolute minimum value of $f(x)$ is
a) 0 b) -15 c) $3 - 2\pi$ d) None of these

96. If the function $f(x) = ax^3 + bx^2 + 11x - 6$ satisfies conditions of Rolle's theorem in $[1, 3]$ for $x = 2 + \frac{1}{\sqrt{3}}$, then value of a and b, respectively, are
a) $-3, 2$ b) $2, -4$ c) $1, 6$ d) None of these

97. Let f be a continuous, differentiable and bijective function. If the tangent to $y = f(x)$ at $x = a$ is also the normal to $y = f(x)$ at $x = b$, then there exists at least one $c \in (a, b)$ such that
a) $f'(c) = 0$ b) $f'(c) > 0$ c) $f'(c) < 0$ d) None of these

98. A function f is defined by $f(x) = |x|^m |x - 1|^n, \forall x \in R$. The local maximum value of the function is $(m, n \in N)$
a) 1 b) $m^n n^m$ c) $\frac{m^m n^n}{(m+n)^{m+n}}$ d) $\frac{(mn)^{mn}}{(m+n)^{m+n}}$

99. Let $f: R \to R$ be a differentiable function for all values of x and has the property that $f(x)$ and $f'(x)$ have opposite signs for all values of x. Then,
a) $f(x)$ is an increasing function b) $f(x)$ is a decreasing function
c) $f^2(x)$ is a decreasing function d) $|f(x)|$ is an increasing function

100. The largest area of a trapezium inscribed in a semi-circle of radius R, if the lower base is on the diameter, is
a) $\frac{3\sqrt{3}}{4}R^2$ b) $\frac{\sqrt{3}}{2}R^2$ c) $\frac{3\sqrt{3}}{8}R^2$ d) R^2

101. A factory D is to be connected by a road with a straight railway line on which a town A is situated. The distance DB of the factory to the railway line is $5\sqrt{3}$ km. Length AB of the railway line is 20 km. Freight charges on the road are twice the charges on the railway. The point $P(AP < AB)$ on the railway line should the road DP be connected so as to ensure minimum freight charges from the factory to the town is
a) $BP = 5$ km b) $AP = 5$ km c) $BP = 7.5$ km d) None of these

102. If f be a continuous function on $[0, 1]$, differentiable in $(0, 1)$ such that $f(1) = 0$, then there exists some $c \in (0, 1)$ such that
a) $cf'(c) - f(c) = 0$ b) $f'(c) + cf(c) = 0$ c) $f'(c) - cf(c) = 0$ d) $cf'(c) + f(c) = 0$

103. The function $f(x)=\frac{x}{2}+\frac{2}{x}$ has a local minimum at

a) $x=2$ b) $x=-2$ c) $x=0$ d) $x=1$

104. The least perimeter of an isosceles triangle in which a circle of radius r can be inscribed is

a) $4\sqrt{3}r$ b) $2\sqrt{3}r$ c) $6\sqrt{3}r$ d) $8\sqrt{3}r$

105. If $f(x)=\begin{cases}\sin^{-1}(\sin x), & x>0\\ \frac{\pi}{2}, & x=0,\\ \cos^{-1}(\cos x), & x<0\end{cases}$, then

a) $x=0$ is a point of maxima
b) $x=0$ is a point of minima
c) $x=0$ is a point of intersection
d) None of these

106. In the interval [0, 1], the function $x^{25}(1-x)^{75}$ takes its maximum value at the point

a) 0 b) $\frac{1}{4}$ c) $\frac{1}{2}$ d) $\frac{1}{3}$

107. If $f(x)=a\log|x|+bx^2+x$ has its extremum values a $x=-1$ and $x=2$, then

a) $a=2, b=-1$ b) $a=2, b=-1/2$ c) $a=-2, b=1/2$ d) None of these

108. A bell tent consists of a conical portion above a cylindrical portion near the ground. For a given volume and a circular base of a given radius, the amount of the canvas used is a minimum when the semi-vertical angle of the cone is

a) $\cos^{-1}2/3$ b) $\sin^{-1}2/3$ c) $\cos^{-1}1/3$ d) None of these

109. A box, constructed from a rectangular metal sheet, is 21 cm by 16 cm by cutting equal squares of sides x from the corners of the sheet and then turning up the projected portions. The value of x so that volume of the box is maximum is

a) 1 b) 2 c) 3 d) 4

110. The maximum value of the function $f(x)=\sin\left(x+\frac{\pi}{6}\right)+\cos\left(x+\frac{\pi}{6}\right)$ in the interval $\left(0,\frac{\pi}{2}\right)$ occurs at

a) $\frac{\pi}{12}$ b) $\frac{\pi}{6}$ c) $\frac{\pi}{4}$ d) $\frac{\pi}{3}$

111. If the function $f(x)$ increases in the interval (a,b), and $\phi(x)=[f(x)]^2$, then

a) $\phi(x)$ increases in (a,b)
b) $\phi(x)$ decreases in (a,b)
c) We cannot say that $\phi(x)$ increases or decreases in (a,b)
d) None of these

112. The maximum value of $(\log x)/x$ is

a) 1 b) $2/e$ c) e d) $1/e$

113. Suppose that f is a polynomial of degree 3 and that $f''(x)\neq 0$ at any of the stationary point. Then

a) f has exactly one stationary point
b) f must have no stationary point
c) f must have exactly two stationary points
d) f has either zero or two stationary point

114. A cylindrical gas container is closed at the top and open at the bottom, if the iron plate of the top is 5/4 times as thick as the plate forming the cylindrical sides, the ratio of the radius to the height of the cylinder using minimum material for the same capacity is

a) 3:4 b) 5:6 c) 4:5 d) None of these

115. $f(x)=(x-1)|(x-2)(x-3)|$, then 'f' decreases in

a) $\left(2-\frac{1}{\sqrt{3}},2\right)$ b) $\left(2,2+\frac{1}{\sqrt{3}}\right)$ c) $\left(2+\frac{1}{\sqrt{3}},4\right)$ d) $(3,\infty)$

116. The minimum value of $2^{(x^2-3)^3+27}$ is

a) 2^{27} b) 2 c) 1 d) None of these

117. The normal to the curve $2x^2+y^2=12$ at the point (2, 2) cuts the curve again at

a) $\left(-\frac{22}{9},-\frac{2}{9}\right)$ b) $\left(\frac{22}{9},\frac{2}{9}\right)$ c) $(-2,-2)$ d) None of these

118. Let $f(x)$ be a function defined as below: $f(x)=\sin(x^2-3x), x\le 0$; and $6x+5x^2, x>0$ Then at $x=0, f(x)$

a) Has a local maximum b) Has a local minimum c) Is discontinuous d) None of these

119. If $f'(x)=|x|-\{x\}$ where $\{x\}$ denotes the fractional part of x, then $f(x)$ is decreasing in

a) $\left(-\frac{1}{2},0\right)$ b) $\left(-\frac{1}{2},2\right)$ c) $\left(-\frac{1}{2},2\right)$ d) $\left(\frac{1}{2},\infty\right)$

120. A differentiable function $f(x)$ has a relative minimum at $x=0$, then the function $y=f(x)+ax+b$ has a relative minimum at $x=0$ for

a) All a and all b b) All b if $a=0$ c) All $b>0$ d) All $a>0$

121. The number of points in the rectangle $\{(x,y)|-12\le x\le 12$ and $-3\le y\le 3\}$ which lie on the curve $y=x+\sin x$ and at which the tangent to the curve is parallel to the x- axis is

a) 0 b) 2 c) 4 d) 8

122. The greatest value of $f(x)=(x+1)^{1/3}-(x-1)^{1/3}$ on $[0,1]$ is

a) 1 b) 2 c) 3 d) $\frac{1}{3}$

123. Consider the function $f:(-\infty,\infty)\to(-\infty,\infty)$ defined by $f(x)=\frac{x^2-a}{x^2+a}, a>0$. Which of the following is not true?

a) Maximum value of f is not attained even though f is bounded

b) $f(x)$ is increasing on $(0,\infty)$ and has minimum at $x=0$

c) $f(x)$ is decreasing on $(-\infty,0)$ and has minimum at $x=0$

d) $f(x)$ is increasing on $(-\infty,\infty)$ and has neither a local maximum nor a local minimum at $x=0$

124. The fuel charges for running a train are proportional to the square of the speed generated in km per hour, and the cost is Rs. 48 at 16 km per hour. If the fixed charges amount to Rs. 300 per hour, the most economical speed is

a) 60 kmph b) 40 kmph c) 48 kmph d) 36 kmph

125. A lamp of negligible height is placed on the ground ℓ_1 away from a wall. A man ℓ_2 m tall is walking at a speed of $\frac{\ell_1}{10}$ m/s from the lamp to the nearest point on the wall. When he is midway between the lamp and the wall, the rate of change in the length of this shadow on the wall is

a) $-\frac{5\ell_2}{2}$ m/s b) $-\frac{2\ell_2}{5}$ m/s c) $-\frac{\ell_2}{2}$ m/s d) $-\frac{\ell_2}{5}$ m/s

126. If $f:R\to R$ and $g:R\to R$ are two functions such that $f(x)+f''(x)=-xg(x)f'(x)$ and $g(x)>0\ \forall\ x\in R$, then the functions $f^2(x)+\left(f'(x)\right)^2$ has

a) A maxima at $x=0$ b) A minima at $x=0$

c) A point of inflexion at $x=0$ d) None of these

127. Given that $f'(x)>g'(x)$ for all real x, and $f(0)=g(0)$, then $f(x)<g(x)$ for all x belongs to

a) $(0,\infty)$ b) $(-\infty,0)$ c) $(-\infty,\infty)$ d) None of these

128. The global maximum value of $f(x)=\log_{10}(4x^3-12x^2+11x-3), x\in[2,3]$ is

a) $-\frac{3}{2}\log_{10}3$ b) $1+\log_{10}3$ c) $\log_{10}3$ d) $\frac{3}{2}\log_{10}3$

129. A functiong(x) is defined as $g(x)=\frac{1}{4}f(2x^2-1)+\frac{1}{2}f(1-x)^2$ and $f'(x)$ is an increasing function, then $g(x)$ is increasing in the interval

a) $(-1,1)$ b) $\left(-\sqrt{\frac{2}{3}},0\right)\cup\left(\sqrt{\frac{2}{3}},\infty\right)$

c) $\left(-\sqrt{\frac{2}{3}},\sqrt{\frac{2}{3}}\right)$ d) None of these

130. Let $f(x)=\begin{cases}-x^2, & \text{for } x<0\\ x^2+8, & \text{for } x\ge 0\end{cases}$. Then x-intercept of the line, that is, the tangent to the graph of $f(x)$ is

a) Zero b) -1 c) -2 d) -4

131. $f(x)=4\tan x-\tan^2 x+\tan^3 x, x\ne n\pi+\frac{\pi}{2}$

a) Is monotonically increasing
b) Is monotonically decreasing
c) Has a point of maxima
d) Has a point of minima

132. Let $f(x) = ax^3 + bx^2 + cx + 1$ have extrema at $x = \alpha, \beta$ such that $\alpha\beta < 0$and $f(\alpha)f(\beta) < 0$. Then the equation $f(x) = 0$ has

a) Three equal real roots
b) One negative roots if $f(\alpha) < 0$ and $f(\beta) > 0$
c) One positive roots if $f(\alpha) > 0$ and $f(\beta) < 0$
d) None of these

133. If $f(x) = \int_{x^2}^{x^2+1} e^{-t^2} dt$, then $f(x)$ increases in

a) $(2, 2)$
b) No value of x
c) $(0, \infty)$
d) $(-\infty, 0)$

134. The number of tangents to the curve $x^{3/2} + y^{3/2} = 2a^{3/2}, a > 0$, which are equally inclined to the axes, is

a) 2
b) 1
c) 0
d) 4

135. At the point $P(a, a^n)$ on the graph of $y = x^n$ $(n \in N)$ in the first quadrant, a normal is drawn. The normal intersects the y-axis at the point $(0, b)$. If $\lim_{a \to 0} b = \frac{1}{2}$, then n equals

a) 1
b) 2
c) 3
d) 4

136. The curves $4x^2 + 9y^2 = 72$ and $x^2 - y^2 = 5$ at $(3, 2)$

a) Touch each other
b) Cut orthogonally
c) Intersect at 45°
d) Intersect at 60°

137. AB is a diameter of a circle and C is any point on the circumference of the circle. Then

a) The area of ΔABC is maximum when it is isosceles
b) The area of ΔABC is minimum when it is isosceles
c) The perimeter of ΔABC is minimum when it is isosceles
d) None of these

138. The equation of the tangent to the curve $y = be^{-x/a}$ at the point where it crosses the y-axis is

a) $\frac{x}{a} - \frac{y}{b} = 1$
b) $ax + by = 1$
c) $ax - by = 1$
d) $\frac{x}{a} + \frac{y}{b} = 1$

139. Let $f(x)$ and $g(x)$ be differentiable for $0 \le x \le 1$, such that $f(0) = 0, g(0) = 0,\ f(1) = 6$. Let there exists a real number c in $(0, 1)$ such that $f'(c) = 2g'(c)$, then the value of $g(1)$ must be

a) 1
b) 3
c) -2
d) -1

140. A curve is represented by the equations $x = \sec^2 t$ and $y = \cot t$, where t is a parameter. If the tangent at the point P on the curve, where $t = \pi/4$, meets the curve again at the point Q, then $|PQ|$ is equal to

a) $\frac{5\sqrt{3}}{2}$
b) $\frac{5\sqrt{5}}{2}$
c) $\frac{2\sqrt{5}}{3}$
d) $\frac{3\sqrt{5}}{2}$

141. If the function $f(x) = \frac{t+3x-x^2}{x-4}$, where t is a parameter that has a minimum and maximum, then the range of values of t is

a) $(0, 4)$
b) $(0, \infty)$
c) $(-\infty, 4)$
d) $(4, \infty)$

142. In which of the following functions is Rolle's theorem applicable?

a) $f(x) = \begin{cases} x, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$ on $[0, 1]$

b) $f(x) = \begin{cases} \frac{\sin x}{x}, & -\pi \le x < 0 \\ 0, & x = 0 \end{cases}$ on $[-\pi, 0]$

c) $f(x) = \frac{x^2-x-6}{x-1}$ on $[-2, 3]$

d) $f(x) = \begin{cases} \frac{x^3-2x^2-5x+6}{x-1} & \text{if } x \ne 1 \text{ on } [-2, 3] \\ -6, & \text{if } x = 1 \end{cases}$

143. A point on the parabola $y^2 = 18x$ at which the ordinate increases at twice the rate of the abscissa is

a) $(2, 6)$
b) $(2, -6)$
c) $\left(\frac{9}{8}, -\frac{9}{2}\right)$
d) $\left(\frac{9}{8}, \frac{9}{2}\right)$

144. At what points of curve $y = \frac{2}{3}x^3 + \frac{1}{2}x^2$, the tangent makes the equal angle with the axis?

a) $\left(\frac{1}{2}, \frac{5}{24}\right)$ and $\left(-1, -\frac{1}{6}\right)$
b) $\left(\frac{1}{2}, \frac{4}{9}\right)$ and $(-1, 0)$
c) $\left(\frac{1}{3}, \frac{1}{7}\right)$ and $\left(-3, \frac{1}{2}\right)$
d) $\left(\frac{1}{3}, \frac{4}{47}\right)$ and$\left(-1, -\frac{1}{3}\right)$

145. $f(x) = (x-2)|x-3|$ is monotonically increasing in

a) $(-\infty, 5/2) \cup (3, \infty)$

b) $(5/2, \infty)$
c) $(2, \infty)$
d) $(-\infty, 3)$

146. A wire of length a is cut into parts which are bent, respectively, in the form of a square and a circle. The least value of the sum of the areas so formed is
a) $\frac{a^2}{\pi + 4}$ b) $\frac{a}{\pi + 4}$ c) $\frac{a}{4(\pi + 4)}$ d) $\frac{a^2}{4(\pi + 4)}$

147. The rate of change of the volume of a sphere w.r.t. its surface area, when the radius is 2 cm, is
a) 1 b) 2 c) 3 d) 4

148. A function is matched below against an interval where it is supposed to be increasing. Which of the following parts is incorrectly matched?
Interval **Function**
a) $[2, \infty) 2x^3 - 3x^2 - 12x + 6$ b) $(-\infty, \infty) x^3 - 3x^2 + 3x + 3$
c) $(-\infty, -4] x^3 + 6x^2 + 6$ d) $\left(-\infty, \frac{1}{3}\right] 3x^2 - 2x + 1$

149. For all $x \in (0, 1)$
a) $e^x < 1 + x$ b) $\log_e(1 + x) < x$ c) $\sin x > x$ d) $\log_e x > x$

150. The maximum value of the function $f(x) = \frac{(1+x)^{0.6}}{1+x^{0.6}}$ in the interval $[0, 1]$ is
a) $2^{0.4}$ b) $2^{-0.4}$ c) 1 d) $2^{0.6}$

151. The real number x when added to its inverse gives the minimum value of the sum at x equals to
a) 1 b) -1 c) -2 d) 2

152. Suppose that f is differentiable for all x and that $f'(x) \leq 2$ for all x. If $f(1) = 2$ and $f(4) = 8$, then $f(2)$ has the value equal to
a) 3 b) 4 c) 6 d) 8

153. The least natural number a for which $x + ax^{-2} > 2, \forall \in (0, \infty)$ is
a) 1 b) 2 c) 5 d) None of these

154. The set of value(s) of a for which the function $f(x) = \frac{ax^3}{3} + (a + 2)x^2 + (a - 1)x + 2$ possesses a negative point of inflection is
a) $(-\infty, -2) \cup (0, \infty)$ b) $\{-4/5\}$ c) $(-2, 0)$ d) Empty set

155. The number of real roots of the equation $e^{x-1} + x - 2 = 0$ is
a) 1 b) 2 c) 3 d) 4

156. Given $g(x) = \frac{x+2}{x-1}$ and the line $3x + y - 10 = 0$, then the line is
a) Tangent to $g(x)$ b) Normal to $g(x)$ c) Chord of $g(x)$ d) None of these

157. If $f(x) = \frac{x}{\sin x}$ and $g(x) = \frac{x}{\tan x}$, where $0 < x \leq 1$, then in this interval
a) Both $f(x)$ and $g(x)$ are increasing function
b) Both $f(x)$ and $g(x)$ are decreasing function
c) $f(x)$ is an increasing function
d) $g(x)$ is an increasing function

158. If for a function $f(x), f'(a) = 0, f''(a) = 0, f'''(a) > 0$, then at $x = a, f(x)$ is
a) Minimum b) Maximum c) Not an extreme point d) Extreme point

159. A function $y = f(x)$ has a second-order derivative $f''(x) = 6(x - 1)$. If its graph passes through the point $(2, 1)$ and at that point tangent to the graph is $y = 3x - 5$, then the value of $f(0)$ is
a) 1 b) -1 c) 2 d) 0

160. Function $f(x) = |x| - |x - 1|$ is monotonically increasing when
a) $x < 0$ b) $x > 1$ c) $x < 1$ d) $0 < x < 1$

161. The greatest value of the function $f(x) = \frac{\sin 2x}{\sin\left(x + \frac{\pi}{4}\right)}$ on the interval $\left(0, \frac{\pi}{2}\right)$ is
a) $\frac{1}{\sqrt{2}}$ b) $\sqrt{2}$ c) 1 d) $-\sqrt{2}$

162. The angle of intersection of the normals at the point $\left(-\frac{5}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$ of the curves $x^2 - y^2 = 8$ and $9x^2 + 25y^2 = 225$ is

a) 0 b) $\frac{\pi}{2}$ c) $\frac{\pi}{3}$ d) $\frac{\pi}{4}$

163. Let $h(x) = x^{m/n}$ for $x \in R$, where m and n are odd numbers and $0 < m < n$, then $y = h(x)$ has

a) No local extremums b) One local maximum c) One local minimum d) None of these

164. Let $f(x)$ be a twice differentiable function for all real values of x and satisfies $f(1) = 1, f(2) = 4, f(3) = 9$ Then which of the following is definitely true?

a) $f''(x) = 2$, for $\forall\ x \in (1,3)$
b) $f''(x) = f'(x) = 5$, for some $x \in (2,3)$
c) $f''(x) = 3\ \forall\ x \in (2,3)$
d) $f''(x) = 2$ for some $x \in (1,3)$

165. If $f(x) = xe^{x(x-1)}$, then $f(x)$ is

a) Increasing on $[-1/2,1]$
b) Decreasing on R
c) Increasing on R
d) Decreasing on $[-1/2,1]$

166. A man is moving away from a tower 41.6 m high at a rate of 2 m/s. If the eye level of the man is 1.6 m above the ground, then the rate at which the angle of elevation of the top of the tower changes, when he is at a distance of 30 m from the foot of the tower, is

a) $-\frac{4}{125}$ radian/s b) $-\frac{2}{25}$ radian/s c) $-\frac{1}{625}$ radian/s d) None of these

167. Let $f(x) = x\sqrt{4ax - x^2}, (a > 0)$. Then $f(x)$ is

a) Increasing in $(0, 3a)$, decreasing in $(-\infty, 0) \cup (3a, \infty)$
b) Increasing in $(a, 4a)$, decreasing in $(5a, \infty)$
c) Increasing in $(0, 4a)$, decreasing in $(-\infty, 0)$
d) None of these

168. The slope of the tangent to the curve $y = f(x)$ at $[x, f(x)]$ is $2x + 1$. If the curve passes through the point $(1, 2)$, then the area bounded by the curve, the x-axis and the line $x = 1$ is

a) $\frac{5}{6}$ b) $\frac{6}{5}$ c) $\frac{1}{6}$ d) 6

169. The abscissa if points P and Q on the curve $y = e^x + e^{-x}$ such that tangents at P and Q make $60°$ with the x-axis

a) In $\left(\frac{\sqrt{3}+\sqrt{7}}{7}\right)$ and in $\left(\frac{\sqrt{3}+\sqrt{5}}{2}\right)$
b) In $\left(\frac{\sqrt{3}+\sqrt{7}}{2}\right)$
c) In $\left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)$
d) $\pm$ In $\left(\frac{\sqrt{3}+\sqrt{7}}{2}\right)$

170. The maximum value of $x^4 e^{-x^2}$ is

a) e^2 b) e^{-2} c) $12e^{-2}$ d) $4e^{-2}$

171. The two curves $x = y^2, xy = a^3$ cut orthogonally at a point, then a^2 is equal to

a) $\frac{1}{3}$ b) 3 c) 2 d) $\frac{1}{2}$

172. Let $f: R \to R$ be a differentiable function $\forall\ x \in R$. If the tangent drawn to the curve at any point $x \in (a, b)$ always lies below the curve, then

a) $f'(x) > 0, f''(x) < 0\ \forall\ x \in (a, b)$
b) $f'(x) > 0, f''(x) < 0\ \forall\ x \in (a, b)$
c) $f'(x) > 0, f''(x) > 0\ \forall\ x \in (a, b)$
d) None of these

173. The function $f(x) = (4\sin^2 x - 1)^n (x^2 - x + 1), n \in N$, has a local minimum at $x = \frac{\pi}{6}$, then

a) n is any even number
b) n is an odd number
c) n is odd prime number
d) n is any natural number

174. Consider the function $f(x) = x\cos x - \sin x$, then identify the statement which is correct

a) f is neither odd nor even

b) f is monotonic decreasing at $x = 0$
c) f has a maxima at $x = \pi$
d) f has a minima at $x = -\pi$

Multiple Correct Answers Type

175. If $f(x) = \int_0^x \frac{\sin t}{t} dt, x > 0$, then
a) $f(x)$ has a local maxima at $x = n\pi (n = 2k, k \in I^+)$
b) $f(x)$ has a local minima at $x = n\pi (n = 2k, k \in I^+)$
c) $f(x)$ has neither maxima nor minima at $x = n\pi (n \in I^+)$
d) $f(x)$ has a local maxima at $x = n\pi (n = 2k - 1, k \in I^+)$

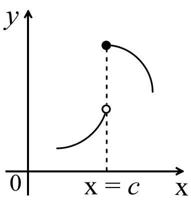
176. The angle between the tangents at any point P and the line joining P to the origin, where P is a point on the curve in $(x^2 + y^2) = c \tan^{-1}\frac{y}{x}$, c is a constant, is
a) Independent of x
b) Independent of y
c) Independent of x but dependent on y
d) Independent of y but dependent on x

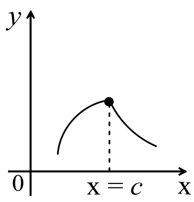
177. Let $g'(x) > 0$ and $f'(x) < 0, \forall x \in R$, then
a) $\big(f(x+1)\big) > g\big(f(x-1)\big)$
b) $f\big(g(x-1)\big) > f\big(g(x+1)\big)$
c) $g\big(f(x+1)\big) < g\big(f(x-1)\big)$
d) $g\big(g(x+1)\big) < g\big(g(x-1)\big)$

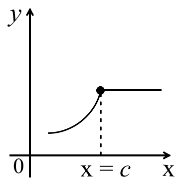
178. If $f(x) = \begin{cases} 3x^2 + 12x - 1, -1 \le x \le 2 \\ 37 - x, 2 < x \le 3 \end{cases}$, then
a) $f(x)$ is increasing in $[-1, 2]$
b) $f(x)$ is continuous on $[-1, 3]$
c) $f'(2)$ does not exist
d) $f(x)$ has the maximum value at $x = 2$

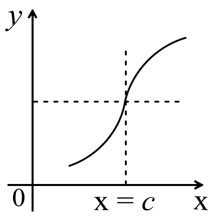
179. Let $f(x) = |x^2 - 3x - 4|, -1 \le x \le 4$, then
a) $f(x)$ is monotonically increasing in $[-1, 3/2]$
b) $f(x)$ is monotonically decreasing in $(3/2, 4)$
c) The maximum value of $f(x)$ is $\frac{25}{4}$
d) The minimum value of $f(x)$ is 0

180. In which of the following graphs is $x = c$ the point of inflection?

a) 

b) 

c) 

d) 

181. The angle formed by the positive y-axis and the tangent to $y = x^2 + 4x - 17$ at $(5/2, -3/4)$ is
a) $\tan^{-1}(9)$
b) $\frac{\pi}{2} - \tan^{-1}(9)$
c) $\frac{\pi}{2} + \tan^{-1}(9)$
d) None of these

182. Let f be a real-valued function defined on the interval$(0, \infty)$, by$f(x) = In\, x + \int_0^x \sqrt{1 + \sin t}\, dt$. Then, which of the following statement (s) is (are) true?
a) $f''(x)$ exist for all $x \in (0, \infty)$
b) $f'(x)$ exists for all $x \in (0, \infty)$ and f' is continuous on $(0, \infty)$, but not differentiable on $(0, \infty)$
c) There exists $\alpha > 1$ such that $|f'(x)| < |f(x)|$ for all $x \in (\alpha, \infty)$
d) There exists $\beta > 0$ such that $|f(x)| + f'(x)| \le \beta$ from all $x \in (0, \infty)$

183. Which of the following function has point of extremum at $x = 0$?
(where $\{x\}$ represents fractional part function)
a) $f(x) = e^{-|x|}$
b) $f(x) = \sin|x|$
c) $f(x) = \begin{cases} x^2 + 4x + 3, x < 0 \\ -x, \quad x \ge 0 \end{cases}$
d) $f(x) = \begin{cases} |x| \;, \quad x < 0 \\ \{x\}, \quad 0 \le x < 1 \end{cases}$

184. Let f and g be increasing and decreasing functions, respectively from $[0, \infty]$ to $[0, \infty]$. Let $h(x) = f\big(g(x)\big)$.

If $h(0) = 0$, then $h(x) - h(1)$ is

a) Always zero b) Always negative c) Always positive d) Strictly increasing

185. If $f(x) = (\sin^2 x - 1)^n$, then $x = \frac{\pi}{2}$ is a point of

a) Local maximum, if n is odd b) Local minimum, if n is odd
c) Local maximum, if n is even d) Local minimum, if n is even

186. In the curve $y = ce^{x/a}$, the

a) Sub-tangent is constant
b) Sub-normal varies as the square of the ordinate
c) Tangent at (x_1, y_1) on the curve intersects the x-axis at a distance of $(x_1 - a)$ from the origin
d) Equation of the normal at the point where the curve cuts y-axis is $cy + ax = c^2$

187. Let $P(x) = a_0 + a_1x^2 + a_2x^4 + \cdots + a_nx^{2n}$ be a polynomial in a real variable x with $0 < a_0 < a_1 < a_2 < \ldots < a_n$. The function $P(x)$ has

a) Neither a maximum nor a minimum
b) Only one maximum
c) Only one minimum
d) Only one maximum and only one minimum

188. If the line $ax + by + c = 0$ is a normal to the curve $xy = 1$, then

a) $a > 0, b > 0$ b) $a > 0, b < 0$ c) $a < 0, b > 0$ d) $a < 0, b < 0$

189. Let $h(x) = f(x) - (f(x))^2 + (f(x))^3$ for every real number x, then

a) h is increasing whenever f is increasing
b) h is increasing whenever f is decreasing
c) h is decreasing whenever f is decreasing
d) Nothing can be said in general

190. For the function $f(x) = \frac{e^x}{1+e^x}$, which of the following hold good?

a) f is monotonic in its entire domain
b) Maximum of f is not attained even though f is bounded
c) f has a point of inflection
d) f has one asymptote

191. If $f(x) = 2x + \cot^{-1} x + \log(\sqrt{1+x^2} - x)$, then $f(x)$

a) Increases in $[0, \infty)$ b) Decreases in $[0, \infty)$
c) Neither increase nor decreases in $[0, \infty)$ d) Increases in $(-\infty, \infty)$

192. The angle between the tangents to the curves $y = x^2$ and $x = y^2$ at $(1, 1)$ is

a) $\cos^{-1}\frac{4}{5}$ b) $\sin^{-1}\frac{3}{5}$ c) $\tan^{-1}\frac{3}{4}$ d) $\tan^{-1}\frac{1}{3}$

193. The critical points of the function $f'(x)$ where $f(x) = \frac{|x-2|}{x^2}$, is

a) 0 b) 2 c) 4 d) 6

194. Let $f(x) = a_5x^5 + a_4x^4 + a_3x^2 + a_1x$, where a_i' s are real and $f(x) = 0$ has a positive root α_0. Then

a) $f'(x) = 0$ has a root α_1 such that $0 < \alpha_1 < \alpha_0$
b) $f'(x) = 0$ has at least two real roots
c) $f''(x) = 0$ has at least one real root
d) None of these

195. The values of parameter a for which the point of minimum of the function $f(x) = 1 + a^2x - x^3$ satisfies the inequality $\frac{x^2+x+2}{x^2+5x+6} < 0$ are

a) $(2\sqrt{3}, 3\sqrt{3})$ b) $(-3\sqrt{3}, -2\sqrt{3})$ c) $(-2\sqrt{3}, 3\sqrt{3})$ d) $(-3\sqrt{2}, 2\sqrt{3})$

196. Let $f(x) = (x-1)^4(x-2)^n, n \in N$. Then $f(x)$ has

a) A maximum at $x = 1$ if n is odd b) A maximum at $x = 1$ if n is even
c) A minimum at $x = 1$ if n is even d) A minima at $x = 2$ if n is even

197. The function $f(x) = -x^2 + \frac{\lambda}{x}$ has a

a) Minimum at $x = 2$ if $\lambda = 16$

b) Maximum at $x = 2$ if $\lambda = 16$

c) Maximum for no real value of λ

d) Point of inflection at $x = 1$ if $\lambda = -1$

198. If $f(x)$ and $g(x)$ are two positive and increasing functions, then which of the following is not always true?

a) $[f(x)]^{g(x)}$ is always increasing

b) If $[f(x)]^{g(x)}$ is decreasing, then $f(x) < 1$

c) If $[f(x)]^{g(x)}$ is increasing, then $f(x) > 1$

d) If $f(x) > 1$, then $[f(x)]^{g(x)}$ is increasing

199. Let $f(x) = \begin{cases} \frac{(x-1)(6x-1)}{2x-1}, & \text{if } x \neq \frac{1}{2} \\ 0, & \text{if } x = \frac{1}{2} \end{cases}$

Then at $x = \frac{1}{2}$, which of the following is/are not true?

a) f has a local maxima

b) f has an local minima

c) f has an inflection point

d) f has a removable discontinuity

200. If $f(x) = x^3 - x^2 + 100x + 2002$, then

a) $f(1000) > f(1001)$

b) $f\left(\frac{1}{2000}\right) > f\left(\frac{1}{2001}\right)$

c) $f(x-1) > f(x-2)$

d) $f(2x-3) > f(2x)$

201. The value of a for which the function $f(x) = (4a-3)(x + \log 5) + 2(a-7)\cot\frac{x}{2}\sin^2\frac{x}{2}$ does not possess critical points is

a) $(-\infty, -4/3)$

b) $(-\infty, -1)$

c) $[1, \infty)$

d) $(2, \infty)$

202. The function $f(x) = \int_{-1}^{x} t(e^t - 1)(t-1)(t-2)^3(t-3)^5 dt$ has a local minimum at $x =$

a) 0

b) 1

c) 2

d) 3

203. For the function $f(x) = x^4(12\log_e x - 7)$

a) The point $(1, -7)$ is the point of inflection

b) $x = e^{1/3}$ is the point of minima

c) The graph is concave downwards in $(0, 1)$

d) The graph is concave upwards in $(1, \infty)$

204. The function $f(x) = x^{1/3}(x-1)$

a) Has two inflection points

b) Has one point of extremum

c) Is non-differentiable

d) Range of $f(x)$ is $[-3 \times 2^{-8/3}, \infty]$

205. Let $f(x) = \begin{cases} x^2 + 3x, -1 \leq x < 0 \\ -\sin x, 0 \leq x < \pi/2 \\ -1 - \cos x, \frac{\pi}{2} \leq x \leq \pi \end{cases}$ then

a) $f(x)$ has global minimum value -2

b) Global maximum value occurs at $x = 0$

c) Global maximum value occurs at $x = \pi$

d) $x = \pi/2$ is point of local minima

206. If composite function $f_1\left(f_2\left(f_3\left(\dots\left(f_n(x)\right)\right)\right)\right)$ $n$times is an increasing function and if r of f_i s are decreasing function while rest are increasing, then maximum value of $r(n-r)$ is

a) $\frac{n^2-1}{4}$, when n is an even number

b) $\frac{n^2}{4}$, when n is an odd number

c) $\frac{n^2-1}{4}$, when n is an odd number

d) $\frac{n^2}{4}$, when n is an even number

207. Which of the following pair(s) of curves is/are orthogonal?

a) $y^2 = 4ax; y = e^{-x/2a}$

b) $y^2 = 4ax;\ x^2 = 4ay$ at $(0, 0)$

c) $xy = a^2; x^2 - y^2 = b^2$

d) $y = ax; x^2 + y^2 = c^2$

208. The smallest positive root of the equation $\tan x - x = 0$ lies in

a) $\left(0, \frac{\pi}{2}\right)$

b) $\left(\frac{\pi}{2}, \pi\right)$

c) $\left(\pi, \frac{3\pi}{2}\right)$

d) $\left(\frac{3\pi}{3}, 2\pi\right)$

209. If $f(x) = \frac{x^2-1}{x^2+1}$, for every real number x, then the minimum value of f

a) Does not exist because f is unbounded
b) Is not attained even through f is unbounded
c) Is equal to 1
d) Is equal to -1

210. The function $y=\frac{2x-1}{x-2}\,(x\neq 2)$
a) Is its own inverse
b) Decreases at all values of x in the domain
c) Has a graph entirely above the x-axis
d) Is unbounded

211. An extremum of the function $f(x)=\frac{2-x}{\pi}\cos\pi\,(x+3)+\frac{1}{\pi^2}\sin\pi\,(x+3), 0<x<4$ occurs at
a) $x=1$
b) $x=2$
c) $x=3$
d) $x=\pi$

212. Let $f(x)=(x^2-1)^n(x^2+x+1)$, then $f(x)$ has local extremum at $x=1$, when
a) $n=2$
b) $n=3$
c) $n=4$
d) $n=6$

213. Let $f(x)=\log(2x-x^2)+\sin\frac{\pi x}{2}$. Then which of the following is/are true?
a) Graph of f is symmetrical about the line $x=1$
b) Maximum value of f is 1
c) Absolute minimum value of f does not exist
d) None of these

214. Which of the following function/functions has/have point of inflection?
a) $f(x)=x^{6/7}$
b) $f(x)=x^6$
c) $f(x)=\cos x+2x$
d) $f(x)=x|x|$

215. The number of values of x where the function $f(x)=\cos x+\cos(\sqrt{2}x)$ attains its maximum is
a) 0
b) 1
c) 2
d) Infinite

216. The equations of the tangents to the curve $y=x^4$ form the point (2, 0) not on the curve are given by
a) $y=0$
b) $y-1=5(x-1)$
c) $y-\frac{4096}{81}=\frac{2048}{27}\left(x-\frac{8}{3}\right)$
d) $y-\frac{32}{243}=\frac{80}{81}\left(x-\frac{2}{3}\right)$

217. If $f'(x)=g(x)(x-a)^2$ where $g(a)\neq 0$ and g is continuous at $x=a$, then
a) f is increasing in the neighbourhood of a if $g(a)>0$
b) f is increasing in the neighbourhood of a if $g(a)<0$
c) f is decreasing in the neighbourhood of a if $g(a)>0$
d) f is decreasing in the neighbourhood of a if $g(a)<0$

218. Let the parabolas $y=x(c-x)$ and $y=x^2+\ ax+b$ touch each other at the point (1, 0), then
a) $a+b+c=0$
b) $a+b=2$
c) $b-c=1$
d) $a+c=-2$

219. The abscissa of a point on the curve $xy=(a+x)^2$, the normal which cuts off numerically equal intercepts from the coordinate axes, is
a) $-\frac{a}{\sqrt{2}}$
b) $\sqrt{2}a$
c) $\frac{a}{\sqrt{2}}$
d) $-\sqrt{2}a$

220. For $F(x)=\int_0^x 2|t|\,dt$, the tangent lines which are parallel to the bisector of the first coordinate angle is
a) $y=x-\frac{1}{4}$
b) $y=x+\frac{1}{4}$
c) $y=x-\frac{3}{2}$
d) $y=x+\frac{3}{2}$

221. Let $f(x)$ be an increasing function defined on $(0,\infty)$. If $f(2a^2+a+1)>f(3a^2-4a+1)$, then the possible integers in the range of a is/are
a) 1
b) 2
c) 3
d) 4

222. The co-ordinates of the point(s) on the graph of the function $f(x)=\frac{x^3}{3}-\frac{5x^2}{2}+7x-4$, where the tangent drawn cuts off intercepts from the co-ordinate axes which are equal in magnitude but opposite in sign, is
a) $(2, 8/3)$
b) $(3, 7/2)$
c) $(1, 5/6)$
d) None of these

223. Let $f(x)=a_5x^5+a_4x^4+a_3x^3+a_2x^2+a_1x$, where $a_i's$ are real and $f(x)=\ 0$ has a positive root α_0. Then
a) $f'(x)=0$ has a root α_1 such that $0<\alpha_1<\alpha_0$
b) $f'(x)=0$ has at least one real root
c) $f''(x)=0$ has at least one real root
d) None of these

224. Points on the curve $f(x)=\frac{x}{1-x^2}$ where the tangent is inclined at an angle of $\frac{\pi}{4}$ to the x-axis are

a) $(0,0)$ b) $\left(\sqrt{3},-\frac{\sqrt{3}}{2}\right)$ c) $\left(-2,\frac{2}{3}\right)$ d) $\left(-\sqrt{3},\frac{\sqrt{3}}{2}\right)$

225. Which of the following is/are correct?

a) Between any two roots of $e^x\cos x=1$, there exists at least one root of $\tan x=1$

b) Between any two roots of $e^x\sin x=1$, there exists at least one root of $\tan x=-1$

c) Between any two roots of $e^x\cos x=1$, there exists at least one root of $e^x\sin x=1$

d) Between any two roots of $e^x\sin x=1$, there exists at least one root of $e^x\cos x=1$

226. If the tangent at any point $P(4m^2,8m^3)$ of $x^3-y^2=0$ is also a normal to the curve $x^3-y^2=0$, then the value of m is

a) $m=\frac{\sqrt{2}}{3}$ b) $m=-\frac{\sqrt{2}}{3}$ c) $m=\frac{3}{\sqrt{2}}$ d) $m=-\frac{3}{\sqrt{2}}$

227. Given $f(x)=4-\left(\frac{1}{2}-x\right)^{2/3}$, $g(x)=\begin{cases}\frac{\tan[x]}{x}, & x\neq 0\\ 1, & x=0\end{cases}$, $h(x)=\{x\}$, $k(x)=5^{\log_2(x+3)}$ then in [0, 1] Lagrange's mean value theorem is NOT applicable to the (where $[\cdot]$ and $\{\cdot\}$ represents greatest integer functions and fractional part functions, respectively)

a) f b) g c) k d) h

228. Which of the following pair(s) of curves is/are orthogonal?

a) $y^2=4ax; y=e^{-x/2a}$ b) $y^2=4ax; x^2=4ay$ at $(0,0)$

c) $xy=a^2; x^2-y^2=b^2$ d) $y=ax; x^2+y^2=c^2$

229. Which of the following hold(s) good for the function $f(x)=2x-3x^{2/3}$?

a) $f(x)$ has two points of extremum

b) $f(x)$ is concave upward for $\forall\, x\in R$

c) $f(x)$ is non-differentiable function

d) $f(x)$ is continuous function

230. In which of the following functions, Rolle's theorem is applicable?

a) $f(x)=|x|$ in $-2\le x\le 2$ b) $f(x)=\tan x$ in $0\le x\le\pi$

c) $f(x)=1+(x-2)^{\frac{2}{3}}$ in $1\le x\le 3$ d) $f(x)=x(x-2)^2$ in $0\le x\le 2$

231. Let $f(x)=2x-\sin x$ and $g(x)=\sqrt[3]{x}$, then

a) Range of gof is R b) gof is one-one

c) Both f and g are one-one d) Both f and g are onto

232. For the cubic function $f(x)=2x^3+9x^2+12x+1$, which one of the following statement/statements hold good?

a) $f(x)$ is non-monotonic

b) $f(x)$ increase in $(-\infty,-2)\cup(-1,\infty)$ and decreases in $(-2,-1)$

c) $f: R\to R$ is bijective

d) Inflection point occurs at $x=-3/2$

233. The function $\frac{\sin(x+a)}{\sin(x+b)}$ has no maxima or minima if

a) $b-a=n\pi, n\in I$ b) $b-a=(2n+1)\pi, n\in I$

c) $b-a=2n\pi, n\in I$ d) None of these

234. Which of the following is true about point of extremum $x=a$ of function $y=f(x)$?

a) At $x=a$, function $y=f(x)$ may be discontinuous

b) At $x=a$, function $y=f(x)$ may be continuous but non-differentiable

c) At $x=a$, function $y=f(x)$ may have point of inflection

d) None of these

235. Let $f(x)=x^4-4x^3+6x^2-4x+1$, then

a) f increases on $[1,\infty)$ b) f decreases on $[1,\infty)$

c) f has a minimum at $x=1$ d) f has neither maximum nor minimum

236. Let $h(x) = f(x) - (f(x))^2 + (f(x))^3$ for every real number x, then

a) h is increasing whenever f is increasing
b) h is increasing whenever f is decreasing
c) h is decreasing whenever f is decreasing
d) h is decreasing whenever f is increasing

237. Which one of the following curves cut the parabola $y^2 = 4ax$ at right angles?

a) $x^2 + y^2 = a^2$
b) $y = e^{-x/2a}$
c) $y = ax$
d) $x^2 = 4ay$

238. Let $f(x) = ax^2 - b|x|$, where a and b are constants. Then at $x = 0, f(x)$ has

a) A maxima whenever $a > 0, b > 0$
b) A maxima whenever $a > 0, b < 0$
c) Minima whenever $a > 0, b < 0$
d) Neither a maxima nor a minima whenever $a > 0, b < 0$

239. Let $f(x) = \sin x + ax + b$, then which of the following is/are true

a) $f(x) = 0$ has only one real root which is positive if $a > 1, b < 0$
b) $f(x) = 0$ has only one real root which is negative if $a > 1, b > 0$
c) $f(x) = 0$ has only one real root which is negative if $a < -1, b < 0$
d) None of these

Assertion - Reasoning Type

This section contain(s) 0 questions numbered 240 to 239. Each question contains STATEMENT 1(Assertion) and STATEMENT 2(Reason). Each question has the 4 choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

a) Statement 1 is True, Statement 2 is True; Statement 2 **is** correct explanation for Statement 1

b) Statement 1 is True, Statement 2 is True; Statement 2 **is not** correct explanation for Statement 1

c) Statement 1 is True, Statement 2 is False

d) Statement 1 is False, Statement 2 is True

240

Statement 1: The function $f(x) = \frac{ae^x + be^{-x}}{ce^x + de^{-x}}$, is increasing function of x, then $bc > ad$

Statement 2: $f'(x) > 0$ for all x

241

Statement 1: Both $f(x) = 2\cos x + 3\sin x$ and $g(x) = \sin^{-1}\frac{x}{\sqrt{13}} - \tan^{-1}\frac{3}{2}$ are increasing for $x \in (0, \pi/2)$

Statement 2: If $f(x)$ is increasing then its inverse is also increasing

242

Statement 1: If $f(x)$ is differentiable in $[0, 1]$ such that $f(0) = f(1) = 0$, then for any $\lambda \in R$, there exists c such that $f'(c) = \lambda f(c), 0 < c < 1$

Statement 2: If $g(x)$ is differentiable in $[0, 1]$, where $g(0) = g(1)$, then there exists c such that $g'(c) = 0, 0 < c < 1$

243

Statement 1: The ordinate of a point describing the circle $x^2 + y^2 = 25$ decreases at the rate of 1.5 cm/s. The rate of change of the abscissa of the point when ordinate equals 4 cm is 2 cm/s

Statement 2: $xdx + ydy = 0$

244 Let $f: R \longrightarrow R$ be a continuous function defined by

$f(x) = \dfrac{1}{e^x + 2e^{-x}}$

Statement 1: $f(c) = \frac{1}{3}$, for some $c \epsilon R$.

Statement 2: $0 < f(x) \leq \dfrac{1}{2\sqrt{2}}$,
for all $x \in R$.

245

Statement 1: Lagrange's mean value theorem is not applicable to $f(x) = |x-1|\,(x-1)$

Statement 2: $|x-1|$ is not differentiable at $x = 1$

246

Statement 1: For all $a, b \in R$ the function $f(x) = 3x^4 - 4x^3 + 6x^2 + ax + b$ has exactly one extremum

Statement 2: If a cubic function is monotonic, then its graph cuts the x-axis only once

247 Observe the following statements
Then which of the following is true?

Statement 1: $f(x) = 2x^3 - 9x^2 + 12x - 3$ is increasing outside the interval $(1, 2)$

Statement 2: $f'(x) < 0$ for $x \in (1, 2)$

248

Statement 1: The points on the curve $y^2 = x + \sin x$ at which the tangent is parallel to x-axis lies on a straight line

Statement 2: Tangent is parallel to x-axis, then $\frac{dy}{dx} = 0$ or $\frac{dx}{dy} = \infty$

249

Statement 1: $f(x) = |x-1| + |x-2| + |x-3|$ has point of minima at $x = 3$

Statement 2: $f(x)$ is non-differentiable at $x = 3$

250

Statement 1: If $f(x) = x(x+3)e^{-x/2}$, then Rolle's theorem applies for $f(x)$ in $[-3, 0]$

Statement 2: LMVT is applied in $f(x) = x(x+3)e^{-x/2}$ in any interval

251

Statement 1: $f(x) = \frac{x^3}{3} + \frac{ax^2}{2} + x + 5$ has positive point of maxima for $a < -2$

Statement 2: $x^2 + ax + 1 = 0$ has both roots positive for $a < -2$

252

Statement 1: If $f'(x) = (x-1)^3(x-2)^8$, then $f(x)$ has neither maximum nor minimum at $x = 2$

Statement 2: $f'(x)$ changes sign from negative to positive at $x = 2$

253

Statement 1: The maximum value of $\left(\sqrt{-3+4x-x^2}+4\right)^2+(x-5)^2$ (where $1 \le x \le 3$) is 36

Statement 2: The maximum distance between the point $(5,-4)$ and the point on the circle $(x-2)^2+y^2=1$ is 6

254 Consider a curve $C: y=\cos^{-1}(2x-1)$ and a straight line $L: 2px-4y+2\pi-p=0$

Statement 1: The set of values of 'p' for which the line L intersects the curve at three distinct points is $[-2\pi,-4]$

Statement 2: The line L is always passing through point of inflection of the curve C

255

Statement 1: If $f(x)$ is continuous in $[a,b]$ and differentiable in (a,b), then there exists at least one $c \in (a,b)$, then $\frac{f(b)-f(a)}{b^3-a^3}=\frac{f'(c)}{3c^2}$

Statement 2: $f'(c)=\dfrac{f(b)-f(a)}{b-a}, \; c \in (a,b)$

256 Let $f(x)=(x^3-6x^2+12x-8)e^x$

Statement 1: $f(x)$ is neither maximum nor minimum at $x=2$

Statement 2: If a function $x=2$ is a point of inflection, then it is not a point of extremum

257

Statement 1: If $f(0)=0, f'(x)=\text{In}(x+\sqrt{1+x^2})$, then $f(x)$ is positive for all $x \in R_0$

Statement 2: $f(x)$ is increasing for $x>0$ and decreasing for $x<0$

258

Statement 1: Both $\sin x$ and $\cos x$ are decreasing functions in $\left(\frac{\pi}{2},\pi\right)$

Statement 2: If a differentiable function decreases in an interval (a,b), then its derivative also decreases in (a,b)

259

Statement 1: The function $f(x)=x^4-8x^3+22x^2-24x+21$ is decreasing for every $x \in (-\infty,1) \cup (2,3)$

Statement 2: $f(x)$ is increasing for $x \in (1,2) \cup (3,\infty)$ and has no point of inflection

260

Statement 1: $f(x)=x+\cos x$ is increasing for $\forall x \in R$

Statement 2: If $f(x)$ is increasing, then $f'(x)$ may vanish at some finite number of points

261

Statement 1: If $g(x)$ is a differentiate function $g(2) \ne 0, g(-2) \ne 0$ and Rolle's theorem is not applicable to $f(x)=\frac{x^2-4}{g(x)}$ in $[-2,2]$, then $g(x)$ has at least one root in $(-2,2)$

Statement 2: If $f(a)=f(b)$, then Rolle's theorem is applicable for $x \in (a,b)$

262

Statement 1: The tangent at $x = 1$ to the curve $y = x^3 - x^2 - x + 2$ again meets the curve at $x = 0$

Statement 2: When the equation of a tangent solved with the given curve, repeated roots are obtained at point of tangency

263 Let $f: R \to R$ is differentiable and strictly increasing function throughout its domain

Statement 1: If $|f(x)|$ is also strictly increasing function, then $f(x) = 0$ has no real roots

Statement 2: When $x \to \infty$ or $\to -\infty, f(x) \to 0$, but cannot be equal to zero

264

Statement 1: The function $f(x) = x \text{ In } x$ is increasing in $(1/e, \infty)$

Statement 2: If both $f(x)$ and $g(x)$ are increasing in (a, b) then $f(x)g(x)$ must be increasing in (a, b)

265

Statement 1: If both functions $f(t)$ and $g(t)$ are continuous on the closed interval $[a, b]$, differentiable on the open interval (a, b), and $g'(t)$ is not zero on that open interval, then there exists some c in (a, b), such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Statement 2: If $f(t)$ and $g(t)$ are continuous and differentiable in $[a, b]$, then there exists some c in (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b-a}$ and $g'(c) = \frac{g(b) - g(a)}{b-a}$ from Lagrange's mean value theorem

266

Statement 1: A tangent parallel to x-axis can be drawn for $f(x) = (x - 1)(x - 2)(x - 3)$ in the interval $[1, 3]$

Statement 2: A horizontal tangent can be drawn in Rolle's theorem

267

Statement 1: Let $f(x) = \sin(\cos x)$ in $\left[0, \frac{\pi}{2}\right]$, then $f(x)$ is decreasing in $\left[0, \frac{\pi}{2}\right]$

Statement 2: $\cos x$ is a decreasing function $\forall\, x \in \left[0, \frac{\pi}{2}\right]$

268 Observe the statement given below

Which of the following is correct?

Statement 1: $f(x) = xe^{-x}$ has the maximum at $x = 1$

Statement 2: $f'(1) = 0$ and $f''(1) < 0$

269

Statement 1: $\alpha^\beta > \beta^\alpha$, for $2.91 < \alpha < \beta$

Statement 2: $f(x) = \frac{\log_e x}{x}$ is a decreasing function for $x > e$

270 Let $y = f(x)$ is a polynomial of degree odd (≥ 3) with real coefficients and (a, b) is any point

Statement 1: There always exists a line passing through (a, b) and touching the curve $y = f(x)$ at some point

Statement 2: A polynomial of degree odd with real coefficients have at least one real root

271

Statement 1: If $27a + 9b + 3c + d = 0$, then the equation $f(x) = 4ax^3 + 3bx^2 + 2cx + d = 0$ has at least one real root lying between (0, 3)

Statement 2: If $f(x)$ is continuous in $[a, b]$, derivable in (a, b) such that $f(a) = f(b)$, then at least one point $c \in (a, b)$ such that $f'(c) = 0$

272

Statement 1: The value of $\left[\lim_{x\to 0+} \frac{\sin x \tan x}{x^2}\right]$ is 1, where $[.]$ denotes the greatest integer function

Statement 2: For $\left(0, \frac{\pi}{2}\right)$, $\sin x < x < \tan x$

273

Statement 1: For the function $f(x) = x^2 + 3x + 2$, LMVT is application in $[1, 2]$ and the value of c is 3/2 because

Statement 2: If LMVT is known to be applicable for any quadratic polynomial in $[a, b]$ then c of LMVT is $(a + b)/2$

274

Statement 1: If Rolle's theorem be applied in $f(x)$, then Lagrange Mean Value Theorem (LMVT) is also applied in $f(x)$

Statement 2: Both Rolle's theorem and LMVT cannot be applied in $f(x) = |\sin|x||$in $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

275

Statement 1: The graph $y = x^3 + ax^2 + bx + c$ has extremum, if $a^2 < 3b$

Statement 2: y is either increasing or decreasing $\forall x \in R$

276

Statement 1: Let $f(x) = 5 - 4\,(x - 2)^{2/3}$, then at $x = 2$ the function $f(x)$ attains neither the least value nor the greatest value

Statement 2: At $x = 2$, first derivative does not exist

277

Statement 1: Let $f: R \to R$ be a function such that$f(x) = x^3 - x^2 + 3x + \sin x$. Then, f is one-one

Statement 2: $f(x)$ is decreasing function

Matrix-Match Type

This section contain(s) 0 question(s). Each question contains Statements given in 2 columns which have to be matched. Statements (A, B, C, D) in **columns I** have to be matched with Statements (p, q, r, s) in **columns II**.

278.

Column-I	Column- II
(A) $f(x) = \|2x - 1\| + \|2x - 3\|$	(p) Has no points of extrema
(B) $f(x) = 2\sin x - x$	(q) Has one point of maxima

(C)	$f(x) = \|x-1\| + \|2x-3\|$	(r)	Has one point of minima
(D)	$f(x) = \|x\| - \|2x-3\|$	(s)	Has infinite points of minima

CODES :

	A	**B**	**C**	**D**
a)	q	r	s	p
b)	s	s	r	q
c)	p	q	r	s
d)	r	p	q	s

279. Consider function $f(x) = x^4 - 14x^2 + 24x - 3$

	Column-I		**Column- II**
(A)	Two negative real roots	(p)	For $p > 120$
(B)	Two real roots of opposite sign	(q)	For $-8 \leq p \leq -5$
(C)	Four real roots	(r)	For $3 < p \leq< 120$
(D)	No real roots	(s)	For $p < -8$ or $-5 < p < 3$

CODES :

	A	**B**	**C**	**D**
a)	r	s	q	p
b)	s	r	p	q
c)	p	q	r	s
d)	q	p	s	r

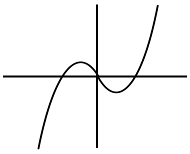
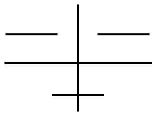
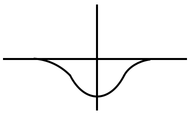
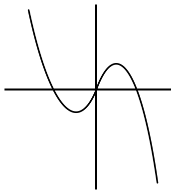
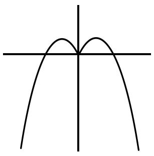
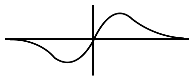
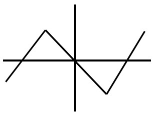
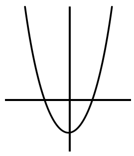
280.

	Column-I		**Column- II**
(A)	The sides of a triangle vary slightly in such a way that its circum-radius remains constant, if $\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} + 1 = \|m\|$, then the values of m is	(p)	1
(B)	The length of sub-tangent to the curve $x^2y^2 = 16$ at the point $(-2, 2)$ is $\|k\|$, then the value of k is	(q)	-1
(C)	The curve $y = 2e^{2x}$ intersects the y-axis at an angle $\cot^{-1}\ \|(8n-4)/3\|$, then the value of n is	(r)	2
(D)	The area of a triangle formed by normal at the point $(1, 0)$ on the curve $x = e^{\sin y}$ with axes is $\|2t+1\|/6$ sq. units, then the value of t is	(s)	-2

CODES :

	A	**B**	**C**	**D**
a)	p, q	r, s	r, q	p, s
b)	r, s	p, q	p, s	r, q
c)	r, q	p, s	p, q	r, s
d)	p, s	r, s	r, q	p, q

281.

	Column-I		**Column- II**
(A)		(p)	
(B)		(q)	
(C)		(r)	
(D)		(s)	

CODES :

	A	**B**	**C**	**D**
a)	r	s	p	q
b)	q	p	r	s
c)	p	q	s	r
d)	s	r	q	p

282. Let $f(x) = (x-1)^m(2-x)^n; m, n \in N$ and $m, n > 2$

	Column-I		**Column- II**
(A)	Both $x = 1$ and $x = 2$ are the points of minima if	(p)	m is even
(B)	$x = 1$ is a point of minima and $x = 2$ is a point of inflection if	(q)	m is odd
(C)	$x = 2$ is a point of minima and $x = 1$ is a point of inflection if	(r)	n is even

(D) Both $x = 1$ and $x = 2$ are the points of inflection if — (s) n is odd

CODES :

	A	B	C	D
a)	P,r	p,s	q,r	q,s
b)	p,s	q,r	p,r	q,s
c)	q,s	p,r	p,s	p,r
d)	q,r	q,s	q,r	p,s

283.

Column-I	Column- II
(A) A circular plate is expanded by heat from radius 6 cm to 6.06 cm. Approximate increase in the area is	(p) 5
(B) If an edge of a cube increases by 2%, then the percentage increase in the volume is	(q) 0.72π
(C) If the rate of decrease of $\frac{x^2}{2} - 2x + 5$ is thrice the rate of decrease of x, then x is equal to (rate of increase is non-zero)	(r) 6
(D) The rate of increase in the area of an equilateral triangle of side 30 cm, when each side increase at the rate of 0.1 cm/s is	(s) $\frac{3\sqrt{3}}{2}$

CODES :

	A	B	C	D
a)	s	q	r	p
b)	q	r	p	s
c)	r	p	s	q
d)	p	s	q	r

284. Match the points on the curve $2y^2 = x + 1$ with the slopes of normals at those points and choose the correct answer.

Column-I	Column- II
(A) $(7, 2)$	(1) $-4\sqrt{2}$
(B) $\left(0, \frac{1}{\sqrt{2}}\right)$	(2) -8
(C) $(1, -1)$	(3) 4
(D) $(3, \sqrt{2})$	(4) 0
	(5) $-2\sqrt{2}$

CODES :

	A	B	C	D
a)	2	4	3	1
b)	2	5	3	1
c)	2	3	5	1
d)	2	5	1	4

285.

Column-I	**Column- II**
(A) $y^2 = 4x$ and $x^2 = 4y$	(p) $90°$
(B) $2y^2 = x^3$ and $y^2 = 32x$	(q) Any one of $\tan^{-1}\frac{3}{4}$ or $\tan^{-1}(16^{\frac{1}{3}})$
(C) $xy = a^2$ and $x^2 + y^2 = 2a^2$	(r) $0°$
(D) $y^2 = x$ and $x^3 + y^3 = 3xy$ at other than origin	(s) $\tan^{-1}\frac{1}{2}$

CODES :

	A	B	C	D
a)	q	p, s	p, q	r
b)	r	p, q	q	p, s
c)	p, q	p, s	r	q
d)	q	r	p, s	p, q

286.

Column-I	**Column- II**
(A) $f(x) = x^2 \log x$	(p) $f(x)$ has one point of minima
(B) $f(x) = x \log_e x$	(q) $f(x)$ has one point of maxima
(C) $f(x) = \frac{\log x}{x}$	(r) $f(x)$ increases in $(0, e)$
(D) $f(x) = x^{-x}$	(s) $f(x)$ decreases in $(0, 1/e)$

CODES :

	A	B	C	D
a)	Q,r	p,s	q	r
b)	p,s	p,s	q,r	q
c)	q	q,r	p,s	p,s
d)	p,s	s	q	p

287. The function $f(x) = \sqrt{(ax^3 + bx^2 + cx + d)}$ has its non-zero local minimum and maximum values at

$x = -2$ and $x = 2$, respectively. If a is a root of $x^2 - x - 6 = 0$, then match the following

	Column-I		Column- II
(A)	The value/values of a	(p)	$= 0$
(B)	The value/values of b	(q)	$= 24$
(C)	The value/values of c	(r)	> 32
(D)	The value/values of d	(s)	-2

CODES :

	A	**B**	**C**	**D**
a)	p	s	r	q
b)	r	q	p	s
c)	s	p	q	r
d)	q	r	s	p

288.

	Column-I		Column- II
(A)	$f(x) = (x-1)^3(x-2)^5$	(p)	Has points of maxima
(B)	$f(x) = 3\sin x + 4\cos x - 5x$	(q)	Has point of minima
(C)	$f(x) = \begin{cases} \sin\frac{\pi x}{2}, 0 < x \le 1 \\ x^2 - 4x + 4, 1 < x < 2 \end{cases}$	(r)	Has point of inflection
(D)	$f(x) = (x-1)^{3/5}$	(s)	Has no point of extrema

CODES :

	A	**B**	**C**	**D**
a)	Q,r	r,s	p,r	r,s
b)	r,s	q,r	p,r	q
c)	p,r	r,s	s	q,r
d)	p	p,r	r,s	p,r

289.

	Column-I		Column- II
(A)	At $x = 1, f(x) = \begin{cases} \log x, x < 1 \\ 2x - x^2, x \ge 1 \end{cases}$	(p)	Is increasing
(B)	At $x = 2, f(x) = \begin{cases} x - 1, x < 2 \\ 0, \quad x = 2 \\ \sin x, x > 2 \end{cases}$	(q)	Is decreasing
(C)	At $x = 0, f(x) = \begin{cases} 2x + 3, x < 0 \\ 5, \quad x = 0 \\ x^2 + 7, x > 0 \end{cases}$	(r)	Has point of maxima

(D) At $x = 0, f(x) = \begin{cases} e^{-x} & x < 0 \\ 0, & x = 0 \\ -\cos x, & x > 0 \end{cases}$ (s) Has point of minima

CODES :

	A	B	C	D
a)	s	r	q	p
b)	r	s	p	q
c)	p	q	r	s
d)	q	p	s	r

290.

Column-I	Column- II
(A) $f(x) = \sin x - x^2 + 1$	(p) Has point of minima
(B) $f(x) = x \log_e x - x + e^{-x}$	(q) Has point of maxima
(C) $f(x) = -x^3 + 2x^2 - 3x + 1$	(r) Is always increasing
(D) $f(x) = \cos \pi x + 10x + 3x^2 + x^3$	(s) Is always decreasing

CODES :

	A	B	C	D
a)	p	q	r	s
b)	r	s	p	q
c)	s	r	q	p
d)	q	p	s	r

Linked Comprehension Type

This section contain(s) 35 paragraph(s) and based upon each paragraph, multiple choice questions have to be answered. Each question has atleast 4 choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Paragraph for Question Nos. 291 to -291

A cubic $f(x) = ax^3 + bx^2 + cx + d$ vanishes at $x = -2$ and has relative minimum/maximum at $x = -1$ and $x = \frac{1}{3}$ and if $\int_{-1}^{1} f(x)\, dx = \frac{14}{3}$

On the basis of above information, answer the following questions

291. The value of c is

a) -2 b) -1 c) 0 d) 2

Paragraph for Question Nos. 292 to - 292

A window of fixed perimeter (including the base of the arch) is in the form of a rectangle surmounted by a semi-circle. The semi-circular portion is fitted with coloured glass, while the rectangular portion is fitted with

clear glass. The clear glass transmits three times as much light per square metre as the coloured glass. Suppose that y is the length and x is the breath of the rectangular portion and P is the perimeter.
On the basis of above information, answer the following questions

292. The ratio of the sides $y: x$ of the rectangle so that the window transmit the maximum light is
a) $3: 2$ b) $6: 6+\pi$ c) $6+\pi: 6$ d) $1: 2$

Paragraph for Question Nos. 293 to - 293

Consider the function $f:(-\infty, \infty) \rightarrow(-\infty, \infty)$ defined by
$f(x)=\frac{x^{2}+a x+1}{x^{2}+a x+1} ; 0<a<2$
On the basis of above information, answer the following questions

293. Which of the following is true?
a) $(2+a)^{2} f''''(1)+(2-a)^{2} f''''(-1)=0$
b) $(2-a)^{2} f''''(1)-(2+a)^{2} f''''(-1)=0$
c) $f''(1) f'(-1)=(2-a)^{2}$
d) $f''(1) f''(-1)=-(2+a)^{2}$

Paragraph for Question Nos. 294 to - 294

Tangent at a point P_{1} [other than (0, 0)] on the curve $y=x^{3}$ meets the curve again at P_{2}. The tangent at P_{2} meets the curve again at P_{3} and so on

294. If P_{1} has co-ordinates (1, 1) then the sum $\lim_{n \rightarrow \infty} \sum_{r=1}^{n} \frac{1}{x_{n}}$ is (where $x_{1}, x_{2}, \ldots$, are abscissas of $P_{1}, P_{2}, \ldots$, respectively)
a) 2/3 b) 1/3 c) 1/2 d) 3/2

Paragraph for Question Nos. 295 to - 295

Consider the curve $x=1-3 t^{2}, y=t-3 t^{3}$. If a tangent at point $\left(1-3 t^{2}, t-3 t^{3}\right)$ inclined at an angle θ to the positive x-axis and another tangent at point $P(-2,2)$ cuts the curve again at Q

295. The value of $\tan \theta+\sec \theta$ is equal to
a) $3 t$ b) t c) $t-t^{2}$ d) $t^{2}-2 t$

Paragraph for Question Nos. 296 to - 296

A spherical balloon is being inflated so that its volume increases uniformly at the rate of 40 $\mathrm{cm}^{3}/\mathrm{min}$

296. At $r=8$, its surface area increases at the rate
a) 8 $\mathrm{cm}^{2}/\mathrm{min}$ b) 10 $\mathrm{cm}^{2}/\mathrm{min}$ c) 20 $\mathrm{cm}^{2}/\mathrm{min}$ d) None of these

Paragraph for Question Nos. 297 to - 297

$f(x) = \sin^{-1} x + x^2 - 3x + \frac{x^3}{3}, x \in [0, 1]$

297. Which of the following is true about $f(x)$?
a) $f(x)$ has a point of maxima
b) $f(x)$ has a point of minima
c) $f(x)$ is increasing
d) $f(x)$ is decreasing

Paragraph for Question Nos. 298 to - 298

Let $f'(\sin x) < 0$ and $f''(\sin x) > 0 \ \forall \ x \in \left(0, \frac{\pi}{2}\right)$ and $g(x) = f(\sin x) + f(\cos x)$

298. Which of the following is true?
a) g' is increasing
b) g' is decreasing
c) g' has a point of minima
d) g' has a point of maxima

Paragraph for Question Nos. 299 to - 299

Consider function $f(x) = \begin{cases} -x^2 + 4x + a, & x \le 3 \\ ax + b, & 3 < x < 4 \\ -\frac{b}{4}x + 6, & x \ge 4 \end{cases}$

(For questions 6 to 8 consider $f(x)$ as a continuous function)

299. Which of the following is true?
a) $f(x)$ is discontinuous function for any value of a and b
b) $f(x)$ is continuous for finite number of values of a and b
c) $f(x)$ cannot be differentiable for any value of a and b
d) $f(x)$ is continuous for infinite values of a and b

Paragraph for Question Nos. 300 to - 300

If $\phi(x)$ is a differentiable real-valued function satisfying $\phi'(x) + 2\phi(x) \le 1$, then it can be adjusted as $e^{2x}\phi'(x) + 2e^{2x}\phi(x) \le e^{2x}$ or $\frac{d}{dx}\left(e^{2x}\phi(x) - \frac{e^{2x}}{2}\right) \le 0$ or $\frac{d}{dx}e^{2x}\left(\phi(x) - \frac{1}{2}\right) \le 0$
Here e^{2x} is called integrating factor which helps in creating single differential coefficient as shown above. Answer the following questions:

300. If $P(1) = 0$ and $\frac{dP(x)}{dx} > P(x)$ for all $x \ge 1$, then
a) $P(x) > 0 \ \forall x > 1$
b) $P(x)$ is a constant function
c) $P(x) < 0 \ \forall x > 1$
d) None of these

Paragraph for Question Nos. 301 to - 301

Let $h(x) = f(x) - a\big(f(x)\big)^2 + a\big(f(x)\big)^3$ for every real number x

301. $h(x)$ increases as $f(x)$ increases for all real values of x if

a) $a \in (0, 3)$ b) $a \in (-2, 2)$ c) $[3, \infty)$ d) None of these

Paragraph for Question Nos. 302 to - 302

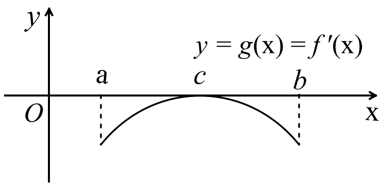
$f(x) = x^3 - 9x^2 + 24x + c = 0$ has three real and distinct root α, β and γ

302. Possible values of c are

a) $(-20, -16)$ b) $(-20, -18)$ c) $(-18, -16)$ d) None of these

Paragraph for Question Nos. 303 to - 303

Consider the graph of $y = g(x) = f'(x)$, given that $f(c) = 0$, where $y = f(x)$ is a polynomial function



303. The graph of $y = f(x)$ will intersect the x-axis

a) Twice b) Once c) Never d) None of these

Paragraph for Question Nos. 304 to - 304

Let $f(x) = 4x^2 - 4ax + a^2 - 2a + 2$ and the global minimum value of $f(x)$ for $x \in [0, 2]$ is equal to 3

304. The number of values of a for which the global minimum value equal to 3 for $x \in [0, 2]$ occurs at the end point if interval [0, 2] is

a) 1 b) 2 c) 3 d) 0

Paragraph for Question Nos. 305 to - 305

Let $f(x) = x^3 - 3(7 - a)x^2 - 3(9 - a^2)x + 2$

305. The values of parameter a if $f(x)$ has a negative point of local minimum are

a) ϕ b) $(-3, 3)$ c) $\left(-\infty, \frac{58}{14}\right)$ d) None of these

Paragraph for Question Nos. 306 to - 306

Consider the function $f(x) = \left(1 + \frac{1}{x}\right)^x$

306. The domain of $f(x)$ is

a) $(-1, 0) \cup (0, \infty)$ b) $R - \{0\}$ c) $(-\infty, -1) \cup (0, \infty)$ d) $(0, \infty)$

Paragraph for Question Nos. 307 to - 307

Consider the function $f(x) = x + \cos x - a$

307. Which of the following is not true about $y = f(x)$?
a) It is an increasing function
b) It is a monotonic function
c) It has infinite points of inflection
d) None of these

Paragraph for Question Nos. 308 to - 308

Consider the function $f(x) = 3x^4 + 4x^3 - 12x^2$

308. $y = f(x)$ increases in the interval
a) $(-1, 0) \cup (2, \infty)$ b) $(-\infty, 0) \cup (1, 2)$ c) $(-2, 0) \cup (1, \infty)$ d) None of these

Paragraph for Question Nos. 309 to - 309

Consider the function $f: R \to R, f(x) = \frac{x^2-6x+4}{x^2+2x+4}$

309. $f(x)$ is
a) Unbounded function b) One-one function c) Onto function d) None of these

Paragraph for Question Nos. 310 to - 310

Consider a polynomial $y = P(x)$ of the least degree passing through $A(-1, 1)$ and whose graph has two points of inflection $B(1, 2)$ and C with abscissa 0 at which the curve is inclined to the positive axis of abscissa at an angle of $\sec^{-1}\sqrt{2}$

310. The value of $P(-1)$ is
a) -1 b) 0 c) 1 d) 2

Paragraph for Question Nos. 311 to - 311

Let $f(x)$ be a real-valued continuous function on R defined as $f(x) = x^2e^{-|x|}$

311. The values of k for which the equation $x^2e^{-|x|} = k$ has four real roots
a) $0 < k < e$ b) $0 < k < \frac{8}{e^2}$ c) $0 < k < \frac{4}{e^2}$ d) None of these

Integer Answer Type

312. At the point $P(a, a^n)$ on the graph of $y = x^n (n \in N)$ in the first quadrant a normal is drawn. The normal

intersects the y-axis at the point $(0, b)$. If $\lim_{a\to 0} b = \frac{1}{2}$, then n equals

313. Let $f(x) = \begin{cases} |x^2 - 3x| + a, 0 \le x < \frac{3}{2} \\ -2x + 3, \quad x \ge \frac{3}{2} \end{cases}$. If $f(x)$ has a local maxima at $x = \frac{3}{2}$, then greatest value of $|4a|$ is

314. Let $f(x)$ be a non-constant thrice differentiable function defined on $(-\infty, \infty)$ such that $f(x) = f(6-x)$ and $f'(0) = 0 = f'(2) = f'(5)$. If n is the minimum number or roots of $(f''(x)^2) + f'(x)f'''(x) = 0$ in the interval $[0, 6]$, then the value of $n/2$ is

315. A right triangle is drawn in a semicircle of radius $\frac{1}{2}$ with one of its legs along the diameter. If the maximum area of the triangle is M, then the value of $32\sqrt{3}M$ is

316. Consider $P(x)$ be a polynomial of degree 5 having extremum at $x = -1$, 1and $\lim_{x\to 0}\left(\frac{P(x)}{x^3} - 2\right) = 4$. Then the value of $[P(1)]$ is (where $[\cdot]$ represents greatest integer function)

317. The number of non-zero integral values of 'a' for which the function $f(x) = x^4 + ax^3 + \frac{3x^2}{2} + 1$ is concave upward along the entire real line is

318. Water is dropped at the rate of $2\text{m}^3/\text{s}$ into a cone of semi-vertical angle 45°. If the rate at which periphery of water surface changes when the height of the water in the cone is 2 m is d, then the value of $5d$ is

319. For a cubic function $y = f(x), f''(x) = 4x$ at each point (x, y) on it and it crosses the x-axis at $(-2, 0)$ at an angle of 45° with positive direction of the x-axis. Then the value of $\left|\frac{f(1)}{5}\right|$ is

320. Number of integral values of b for which the equation $\frac{x^3}{3} - x = b$ has 3 distinct solutions is

321. The least area of a circle circumscribing any right triangle of area $\frac{9}{\pi}$ is

322. If α is an integer satisfying $|\alpha| \le 4 - |[x]|$, where x is a real number for which $2x\tan^{-1}x$ is greater than or equal to $\text{In}(1 + x^2)$, then the number of maximum possible values of a (where $[\cdot]$ represents the greatest integer function)

323. If m is the minimum value of $f(x, y) = x^2 - 4x + y^2 + 6y$ when x and y are subjected to the restrictions $0 \le x \le 1$ and $0 \le y \le 1$, then the value of $|m|$ is

324. Let $y = f(x)$ be drawn with $f(0) = 2$ and for each real number the tangent to $y = f(x)$ at $(a, f(a))$, has x intercept $(a-2)$. If $f(x)$ is of the form of ke^{px}, then $\left(\frac{k}{p}\right)$ has the value equal to

325. Let $f(x) = \begin{cases} x^{3/5} \text{ if } x \le 1 \\ -(x-2)^3 \text{ if } x > 1 \end{cases}$, then the number of critical points on the graph of the function is

326. Suppose a, b, c are such that the curve $y = ax^2 + bx + c$ is tangent to $y = 3x - 3$ at $(1, 0)$ and is also tangent to $y = x + 1$ at $(3, 4)$, then the value of $(2a - b - 4c)$ equals

327. Let $f(x) = \begin{cases} x + 2, x < -1 \\ x^2, -1 \le x < 1 \\ (x-2)^2, x \ge 1 \end{cases}$, then number of times $f'(x)$ changes its sign in $(-\infty, \infty)$ is

328. A curve is given by the equations $x = \sec^2\theta, y = \cot\theta$. If the tangent at P where $\theta = \pi/4$ meets the curve again at Q, then $[PQ]$ is, where $[\cdot]$ represents the greatest integer function

329. From a given solid cone of height H, another inverted cone is carved whose height is h such that its volume is maximum. Then the ratio H/h is

330. There is a point (p, q) on the graph of $f(x) = x^2$ and a point (r, s) on the graph of $\text{g}(x) = \frac{-8}{x}$, where $p > 0$ and $r > 0$. If the line through (p, q) and (r, s) is also tangent to both the curves at these points, respectively, then the value of $p + r$ is

331. Let $f(x)$ be a cubic polynomial which has local maximum at $x = -1$ and $f'(x)$ has a local minimum has at $x = 1$. If $f(-1) = 10$ and $f(3) = -22$, then one fourth of the distance between its two horizontal tangents is

332. A rectangle with one side lying along the x-axis is to be inscribed in the closed region of the xy plane bounded by the lines $y = 0, y = 3x$ and $y = 30 - 2x$. If M is the largest area of such a rectangle, then the value of $\frac{2M}{27}$ is

333. If the slope of line through the origin which is tangent to the curve $y = x^3 + x + 16$ is m, then the value of $m - 4$ is

334. If d is the minimum distance between the curves $f(x) = e^x$ and $g(x) = \log_e x$, then the value of d^6 is

335. The least integral value of x where $f(x) = \log_{1/2}(x^2 - 2x - 3)$ is monotonically decreasing is

336. A curve is defined parametrically by the equations $x = t^2$ and $y = t^3$. A variable pair of perpendicular lines through the origin 'O' meet the curve at P and Q. If the locus of the point of intersection of the tangents at P and Q is $ay^2 = bx - 1$, then the value of $(a + b)$ is

337. Let C be a curve defined by $y = e^{a+bx^2}$. The curve C passes through the point $P(1, 1)$ and the slope of the tangent at P is (-2). Then the value of $2a - 3b$ is

338. Let $f(x) = \begin{cases} |x^3 + x^2 + 3x + \sin x| \left(3 + \sin\frac{1}{x}\right), & x \neq 0, \\ 0 & , x = 0 \end{cases}$ then number of points where $f(x)$ attains its minimum value is

339. If the curve C in the xy plane has the equation $x^2 + xy + y^2 = 1$, then the fourth power of the greatest distance of a point on C from the origin, is

: ANSWER KEY :

1)	c	2)	b	3)	c	4)	b
5)	b	6)	a	7)	d	8)	d
9)	a	10)	a	11)	b	12)	c
13)	b	14)	a	15)	c	16)	c
17)	d	18)	b	19)	a	20)	d
21)	a	22)	a	23)	d	24)	b
25)	c	26)	c	27)	a	28)	b
29)	d	30)	c	31)	b	32)	c
33)	c	34)	d	35)	d	36)	a
37)	d	38)	b	39)	b	40)	d
41)	d	42)	b	43)	c	44)	b
45)	d	46)	a	47)	c	48)	d
49)	d	50)	d	51)	d	52)	d
53)	c	54)	a	55)	c	56)	a
57)	c	58)	d	59)	c	60)	b
61)	a	62)	a	63)	a	64)	b
65)	c	66)	b	67)	a	68)	d
69)	b	70)	d	71)	b	72)	d
73)	c	74)	b	75)	b	76)	d
77)	a	78)	c	79)	a	80)	c
81)	c	82)	b	83)	d	84)	d
85)	b	86)	d	87)	a	88)	b
89)	c	90)	a	91)	b	92)	d
93)	d	94)	b	95)	b	96)	d
97)	a	98)	c	99)	c	100)	a
101)	a	102)	d	103)	a	104)	c
105)	a	106)	b	107)	b	108)	a
109)	c	110)	a	111)	c	112)	d
113)	d	114)	c	115)	a	116)	c
117)	a	118)	b	119)	a	120)	b
121)	a	122)	b	123)	d	124)	b
125)	b	126)	a	127)	b	128)	b
129)	b	130)	b	131)	a	132)	d
133)	d	134)	b	135)	c	136)	b
137)	a	138)	d	139)	b	140)	d
141)	c	142)	d	143)	d	144)	a
145)	a	146)	d	147)	a	148)	d
149)	b	150)	c	151)	a	152)	b
153)	b	154)	a	155)	a	156)	a
157)	c	158)	c	159)	b	160)	d
161)	c	162)	b	163)	a	164)	d
165)	a	166)	a	167)	a	168)	a
169)	b	170)	d	171)	d	172)	c
173)	a	174)	b	1)	b,d	2)	
	a,b	3)	b,c	4)	a,b,c,d		
5)	a,b,c,d	6)	a,b,d	7)	b,c	8)	
	b,c						

9)	a,b,d	10)	a	11)	a,d	12)	
	a,b,c,d						
13)	c	14)	b,c	15)	a,c	16)	
	a,b,c						
17)	a,d	18)	a,b,c	19)	a,b,c	20)	
	a,b,c						
21)	a,b	22)	a,c,d	23)	a,c,d	24)	
	a,b,c						
25)	a,b,d	26)	b,c	27)	a,d	28)	
	b,d						
29)	a,b,c,d	30)	a,b,c,d	31)	a,b,c,d	32)	
	c,d						
33)	a,b,c,d	34)	c	35)	d	36)	
	a,b,d						
37)	a,c	38)	a,c,d	39)	a,b,c	40)	
	c,d						
41)	b	42)	a,c	43)	a,d	44)	
	a,c,d						
45)	a,c	46)	a,b	47)	b,c,d	48)	
	a,b						
49)	a,b,c	50)	a,b,d	51)	a,b,c	52)	
	a,b						
53)	a,b,d	54)	a,b,c,d	55)	a,b,c,d	56)	d
57)	a,b,c,d	58)	a,b,d	59)	a,b,c	60)	
	a,b,c						
61)	a,c	62)	a,c	63)	b,d	64)	
	a,c						
65)	a,b,c	1)	d	2)	a	3)	a
	4)	a					
5)	c	6)	d	7)	a	8)	b
9)	d	10)	d	11)	b	12)	a
13)	c	14)	a	15)	b	16)	b
17)	c	18)	a	19)	c	20)	a
21)	b	22)	c	23)	d	24)	a
25)	c	26)	c	27)	b	28)	b
29)	a	30)	a	31)	a	32)	a
33)	b	34)	a	35)	b	36)	a
37)	d	38)	c	1)	b	2)	a
	3)	a	4)	d			
5)	a	6)	b	7)	b	8)	c
9)	b	10)	c	11)	a	12)	b
13)	d	1)	b	2)	b	3)	a
	4)	a					
5)	a	6)	b	7)	b	8)	a
9)	d	10)	a	11)	a	12)	a
13)	b	14)	b	15)	a	16)	c
17)	d	18)	c	19)	d	20)	c

21)	**c**	**1)**	**2**	**2)**	**9**	**3)**	**6**	**17)**	**3**	**18)**	**3**	**19)**	**5**	**20)**	**8**
	4)	**9**						**21)**	**5**	**22)**	**9**	**23)**	**8**	**24)**	**4**
5)	**2**	**6)**	**4**	**7)**	**5**	**8)**	**3**	**25)**	**7**	**26)**	**5**	**27)**	**1**	**28)**	**4**
9)	**1**	**10)**	**9**	**11)**	**9**	**12)**	**3**								
13)	**4**	**14)**	**3**	**15)**	**9**	**16)**	**4**								

: HINTS AND SOLUTIONS :

1 **(c)**

$\because f'(x) = x^x[1 + \log x] = x^x \log(ex)$

$f'(x) < 0$

$\Rightarrow \log(ex) < 0$

$\Rightarrow 0 < ex < 1$

$\Rightarrow 0 < x < 1/e$

2 **(b)**

Clearly, $f(x)$ in decreasing just before $x = 3$ and increasing after $x = 3$. For $x = 3$ to be the point of local minima, $f(3) \le f(3^-)$

$\Rightarrow -15 \le 12 - 27 + \text{In}(a^2 - 3a + 3)$

$\Rightarrow a^2 - 3a + 3 \ge 1$

$\Rightarrow a \in (-\infty, 1) \cup (2, \infty)$

3 **(c)**

$a^2x^4 + b^2y^4 = c^6$

$\Rightarrow y = \left(\dfrac{c^6 - a^2x^4}{b^2}\right)^{1/4}$

$\Rightarrow f(x) = xy = x\left(\dfrac{c^6 - a^2x^4}{b^2}\right)^{1/4}$

$\Rightarrow f(x) = \left(\dfrac{c^6x^4 - a^2x^8}{b^2}\right)^{1/4}$

Differentiate $f(x)$ w.r.t. x,

$\Rightarrow f'(x) = \dfrac{1}{4}\left(\dfrac{c^6x^4 - a^2x^8}{b^2}\right)^{-3/4}\left(\dfrac{4x^3c^6}{b^2} - \dfrac{8x^7a^2}{b^2}\right)$

Put $f'(x) = 0, \frac{4x^3c^6}{b^2} - \frac{8x^7a^2}{b^2} = 0$

$\Rightarrow x^4 = \dfrac{c^6}{2a^2} \Rightarrow x = \pm\dfrac{c^{3/2}}{2^{1/4}\sqrt{a}}$

At $x = \frac{c^{3/2}}{2^{1/4}\sqrt{a}}$, $f(x)$ will be maximum,

So $f\left(\frac{c^{3/2}}{2^{1/4}\sqrt{a}}\right) = \left(\frac{c^{12}}{2a^2b^2} - \frac{c^{12}}{4a^2b^2}\right)^{1/4} = \left(\frac{c^{12}}{4a^2b^2}\right)^{1/4}$

$= \dfrac{c^2}{\sqrt{2ab}}$

Alternative method:

Since A.M. ≥ G.M.

$\dfrac{a^2x^4 + b^2y^4}{2} \ge \sqrt{a^2x^4b^2y^4}$

$\Rightarrow abx^2y^2 \le \dfrac{c^6}{2}$

$\Rightarrow xy \le \dfrac{c^3}{\sqrt{2ab}}$

Hence, maximum value of xy is $\frac{c^3}{\sqrt{2ab}}$

4 **(b)**

We have $f(x) = \sin^4 x + \cos^4 x = \frac{3}{4} + \frac{1}{4}\cos 4x$

$\Rightarrow f'(x) = -\sin 4x$

Now, $f'(x) > 0 \Rightarrow -\sin 4x > 0 \Rightarrow \sin 4x < 0 \Rightarrow$

$\pi < 4x < 2\pi$

$\Rightarrow \pi/4 < x < \pi/2$

5 **(b)**

$V = x^3$ and the percent error in measuring

$x = \frac{dx}{x} \times 100 = k$

The percent error in measuring volume

$= \frac{dV}{V} \times 100$

Now, $\frac{dV}{dx} = 3x^2$

$\Rightarrow dV = 3x^2dx \Rightarrow \dfrac{dV}{V} = \dfrac{3x^2dx}{x^3} = 3\dfrac{dx}{x}$

$\therefore \dfrac{dV}{V} \times 100 = 3\dfrac{dx}{x} \times 100 = 3k$

6 **(a)**

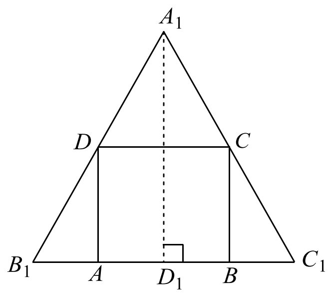
For Lagrange's Mean value theorem we know, $f(x)$ should be continuous in [a, b] and differentiable in] a, b [.

(a) Given, $f(x) = \begin{cases} \dfrac{1}{2} - x, & x < \dfrac{1}{2} \\ \left(\dfrac{1}{2} - x\right)^2, & x \ge \dfrac{1}{2} \end{cases}$

Which is clearly not differentiable at $x = \frac{1}{2}$; *as* RHD at $(x = 1/2) = -1$ *and* LHD at $(x = 1/2) = 0$ ⇒Lagrange's Mean Value is not applicable.

Where option, (b), (c), (d) are continuous and differentiable.

7 **(d)**



Let $BD_1 = x \Rightarrow BC_1 = (a - x)$

$\Rightarrow BC = (a - x)\tan\dfrac{\pi}{3} = \sqrt{3}\,(a - x)$

Now, area of rectangle $ABCD$,

$\Delta = (AB)(BC) = 2\sqrt{3}x(a - x)$

$\Rightarrow \Delta \le 2\sqrt{3}\left(\frac{x+a-x}{2}\right)^2 = \frac{\sqrt{3}a^2}{2}$ (using A.M. ≥ G.M.)

8 **(d)**

Since $f(x) = \frac{K\sin x + 2\cos x}{\sin x + \cos x}$ is increasing for all x, therefore $f'(x) > 0$ for all x

$\Rightarrow \frac{K-2}{(\sin x+\cos x)^2} > 0$ for all x

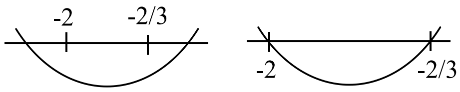
$\Rightarrow K-2>0 \Rightarrow K>2$

9 **(a)**

Here $f'(x) \leq 0$

$\Rightarrow 3x^2+8x+\lambda \leq 0 \ \forall x \in \left(-2,-\frac{2}{3}\right)$

Then situations for $f'(x)$ is as follow:



Given that $f(x)$ decreases in the largest possible interval $\left(-2,-\frac{2}{3}\right)$, then $f'(x)=0$ must have roots -2 and $-2/3$

$\Rightarrow$ Product of roots is $(-2)\left(-\frac{2}{3}\right)=\frac{\lambda}{3} \Rightarrow \lambda=4$

10 **(a)**

Consider the function $f(x)=ax^3+bx^2+cx+d$ on [0, 1] then being a polynomial, it is continuous on [0, 1] and differentiable on (0, 1) and $f(0)=f(1)=d$

$f(0)=d, f(1)=a+b+c+d=d$ [as given $a+b+c=0$]

$\therefore$ By Rolle's theorem, there exists at least one $x \in (0,1)$ such that $f'(x)=0$

$\Rightarrow 3ax^2+2bx+c=0$

Thus, equation $3ax^2+2bx+c=0$ has at least one root in [0, 1]

12 **(c)**

Given, $g(u)=2\tan^{-1}(e^u)-\frac{\pi}{2}, for\ u \in (-\infty,\infty)$

Now, $g(-u)=2\tan^{-1}(e^{-u})-\frac{\pi}{2}$

$=2(\cot^{-1}(e^u))-\frac{\pi}{2}$

$=2\left(\frac{\pi}{2}-\tan^{-1}(e^u)\right)-\frac{\pi}{2}$

$=-g(u)$

$\therefore$ $g(u)$ Is an odd function.

Also, $g'(u)=2\frac{1}{1+(e^u)^2}.e^u-0>0$

Which is strictly increasing in $(-\infty,\infty)$

13 **(b)**

$f'(x)=4-2\sec^2 2x=2(1-\tan^2 2x)$

For the continuous domain $\left(-\frac{\pi}{4},\frac{\pi}{4}\right), f'(x) \geq 0$ in $\left[-\frac{\pi}{8},\frac{\pi}{8}\right]$

and $f'(x) \leq 0$ in $\left(-\frac{\pi}{4},-\frac{\pi}{8}\right] \cup \left[\frac{\pi}{8},\frac{\pi}{4}\right)$

So the required largest continuous interval is $\left[-\frac{\pi}{8},\frac{\pi}{8}\right]$, length $=\frac{\pi}{4}$

14 **(a)**

Putting $x=0$ in the given curve, we obtain $y=1$

So, the given point is (0, 1)

Now, $y=e^{2x}+x^2 \Rightarrow \frac{dy}{dx}=2e^{2x}+2x \Rightarrow$

$\left(\frac{dy}{dx}\right)_{(0,1)}=2$

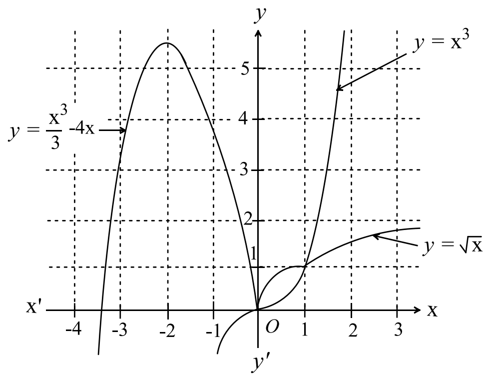
The equation of the tangent at (0, 1) is

$y-1=2(x-0) \Rightarrow 2x-y+1=0$ (1)

Required distance = length of the $\perp$ from (0, 0) on (1) $=\frac{1}{\sqrt{5}}$

15 **(c)**

Function is increasing in $(-\infty,-2) \cup (0,\infty)$, function is decreasing in $(-2,0)$



$x=-2$ is local maxima, $x=0 \rightarrow$ local minima

Derivable $\forall x \in R-\{0,1\}$

Continuous $\forall x \in R$

16 **(c)**

$x=a(\cos\theta+\theta\sin\theta), y=a(\sin\theta-\theta\cos\theta)$

$\frac{dx}{d\theta}=a(-\sin\theta+\sin\theta+\theta\cos\theta)=a\theta\cos\theta$

(1)

$\frac{dy}{d\theta}=a(\cos\theta-\cos\theta+\theta\sin\theta)=a\,\theta\sin\theta$

(2)

$\Rightarrow \frac{dy}{dx}=\tan\theta$ (slope of the tangent)

$\Rightarrow$ Slope of the normal $=-\cot\theta$

$\therefore$ Equation of the normal is

$y-a(\sin\theta-\theta\cos\theta)$

$=-\frac{\cos\theta}{\sin\theta}(x$

$-a(\cos\theta+\theta\sin\theta))$

$\Rightarrow y\sin\theta-a\sin^2\theta+a\theta\sin\theta\cos\theta$

$=-x\cos\theta+a\cos^2\theta+a\theta\sin\theta\cos\theta$

$\Rightarrow x\cos\theta+y\sin\theta=a$

As θ varies, inclination is not constant. Therefore, (a) is not correct

Clearly, it does not pass through (0, 0)

Its distance from the origin $=\left|\frac{a}{\sqrt{\cos^2\theta+\sin^2\theta}}\right|=a$,

Which is a constant

17 **(d)**

Differentiating w.r.t. x, we get $e^y\frac{dy}{dx}=2x$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{2x}{1+x^2} \quad (\because e^y = 1 + x^2)$

$\Rightarrow m = \dfrac{2x}{1+x^2}$ or $|m| = \dfrac{2\,|x|}{1+|x|^2}$

But $1 + |x|^2 - 2|x| = (1-|x|)^2 \geq 0$

$\Rightarrow 1 + |x|^2 \geq 2|x|,$

$\therefore\ |m| \leq 1$

18 **(b)**

Given $A + B = 60° \Rightarrow B = 60° - A$

$\Rightarrow \tan B = \tan(60° - A) = \dfrac{\sqrt{3} - \tan A}{1 + \sqrt{3}\tan A}$

Now $z = \tan A \tan B$

or $z = \frac{t(\sqrt{3}-t)}{1+\sqrt{3}t} = \frac{\sqrt{3}t - t^2}{1+\sqrt{3}t}$

Where $t = \tan A$

$\dfrac{dz}{dt} = -\dfrac{(t+\sqrt{3})(\sqrt{3}t - 1)}{(1+\sqrt{3}t)^2} = 0$

$\Rightarrow t = 1/\sqrt{3}$

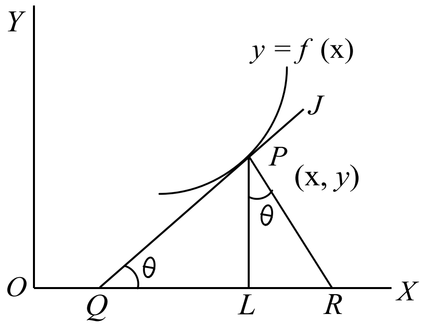
$\Rightarrow t = \tan A = \tan 30°$

The other value of rejected as both A and B are +ve accute angles

If $t < \frac{1}{\sqrt{3}}, \frac{dz}{dt}$ = positive and if $t > \frac{1}{\sqrt{3}}, \frac{dz}{dt} = -$ve

Hence maximum when $t = \frac{1}{\sqrt{3}}$ and maximum value $= \frac{1}{3}$

19 **(a)**



Given curve is $2x^2y^2 - x^4 = c$ (1)

Sub-normal at $P(x, y) = y\frac{dy}{dx}$ (2)

From (1), we get $2\left(x^2 2y\frac{dy}{dx} + 2xy^2\right) - 4x^3 = 0$

$\Rightarrow \frac{dy}{dx} = \frac{x\,(x^2 - y^2)}{x^2 y}$ (3)

Now, $x\,(x - yy') = x^2 - xy\frac{dy}{dx}$

$= x^2 - (x^2 - y^2)$ [from (3)]

$= y^2$

$\Rightarrow$ Mean proportion $= \sqrt{x\,(x - yy')} = y$

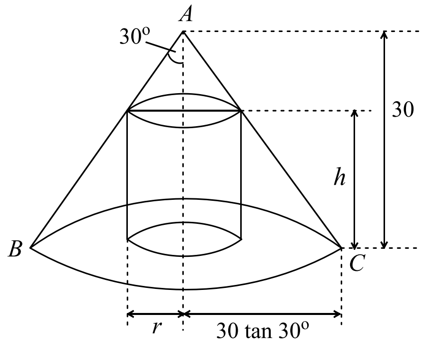
20 **(d)**

$\frac{dy}{dx} = ke^{kx} = k$ at (0, 1). Equation of the tangent is $y - 1 = kx$

Point of intersection with x-axis is $x = -\frac{1}{k}$, where

$-2 \leq -\dfrac{1}{k} \leq -1 \Rightarrow k \in \left[\dfrac{1}{2}, 1\right]$

21 **(a)**



From geometry, we have $\frac{r}{30\tan 30°} = \frac{30-h}{30}$

$\Rightarrow h = 30 - \sqrt{3}r$

Now, the volume of cylinder, $V = \pi r^2 h = \pi r^2(30 - \sqrt{3}r)$

Now, let $\frac{dV}{dr} = 0 \Rightarrow \pi(60r - 3\sqrt{3}r^2) = 0 \Rightarrow r = \frac{20}{\sqrt{3}}$

Hence, $V_{\max} = \pi\left(\frac{20}{\sqrt{3}}\right)^2\left(30 - \sqrt{3}\frac{20}{\sqrt{3}}\right) = \pi\frac{400}{3} \times 10$

$= \dfrac{4000\pi}{3}$

22 **(a)**

$\dfrac{dx}{dy} = a + \dfrac{a}{2}\,2\cos 2t = a\,[1 + \cos 2t] = 2a\cos^2 t$

and $\frac{dy}{dt} = 2a\,(1 + \sin t)\cos t$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{2a\,(1+\sin t)\cos t}{2a\cos^2 t} = \dfrac{(1+\ \sin t)}{\cos t}$

Then, the slope of the tangent

$\tan\theta = \dfrac{(\cos(t/2)\ + \sin(t/2))^2}{\cos^2(t/2) - \sin^2(t/2)}$

$= \dfrac{1 + \tan\frac{t}{2}}{1 - \tan\frac{t}{2}} = \tan\left(\dfrac{\pi}{4} + \dfrac{t}{2}\right)$

$\Rightarrow \theta = \dfrac{\pi\ + 2t}{4}$

23 **(d)**

If $f(x)$ has an extremum at $x = \pi/3$, then $f'(x) = 0$ at $x = \pi/3$

Now, $f(x) = a\sin x + \frac{1}{3}\sin 3x$

$\Rightarrow f'(x) = a\cos x + \cos 3x$

$f'(\pi/3) = 0$

$\Rightarrow a\cos(\pi/3) + \cos\pi = 0$

$\Rightarrow a = 2$

24 **(b)**

We must have $\log_{1/3}(\log_3(\sin x + a)) < 0\ \forall x \in R$

$\Rightarrow \log_3(\sin x + a) > 1\ \forall x \in R$

$\Rightarrow \sin x + a > 3\ \ \forall x \in R$

$\Rightarrow a > 3 - \sin x\ \ \forall x \in R$

$\Rightarrow a > 4$

25 **(c)**

$f(0) = -1$; $f(1) = 7$. So $f(0)$ and $f(1)$ have opposite sign

26 **(c)**

$y = x^2 + bx - b \Rightarrow \frac{dy}{dx} = 2x + b$

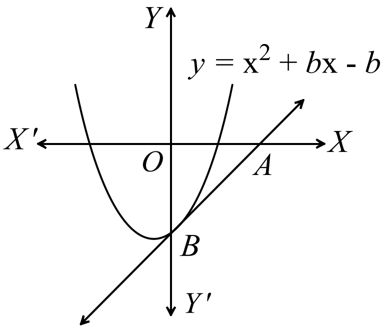
$\Rightarrow$ Equation of the tangent at (1, 1) is

$y - 1 = (2 + b)(x - 1)$

$\Rightarrow (b + 2)x - y = b + 1$

x-intercept $= \frac{b+1}{b+2} = OA$

and y-intercept $= -(b + 1) = OB$



Given area of triangle OAB is $=2$

$\Rightarrow \frac{1}{2} OA \times OB = 2$

$\Rightarrow \frac{1}{2}\left(\frac{b+1}{b+2}\right)[-(b+1)] = 2$

$\Rightarrow b^2 + 2b + 1 = -4(b + 2)$

$\Rightarrow b^2 + 6b + 9 = 0$

$\Rightarrow (b + 3)^2 = 0 \Rightarrow b = -3$

27 **(a)**

Here, $f(x) = x^3 + bx^2 + cx + d$

$\Rightarrow f'(x) = 3x^2 + 2bx + c$

(As we know, if $ax^2 + bx + c > 0$ for all x

$\Rightarrow a > 0$ *and* $D < 0$)

Now, $D = 4b^2 - 12c = 4(b^2 - c) - 8c$

(where$b^2 - c < 0$ *and* $c > 0$)

$\therefore D = (-\text{ve})$ *or* $D < 0$.

$\Rightarrow f'(x) = 3x^2 + 2bx + c > 0$for all $x \in (-\infty, \infty)$

Hence, $f(x)$is strictly increasing function.

28 **(b)**

$\frac{dV}{dt} = -4\text{ cm}^3/\text{min}; \frac{dS}{dt} = ?$ when $V = 125\text{ cm}^2$

$V = x^3; S = 6x^2 \Rightarrow \frac{dV}{dt} = 3x^2\frac{dx}{dt}$

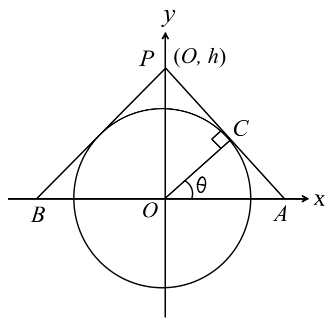
$-4 = 3x^2\frac{dx}{dt}$ (1)

Also $\frac{dS}{dt} = 12x\frac{dx}{dt}$

$\frac{dS}{dt} = -\frac{16}{x}$; when $V = 125 = x^3 \Rightarrow x = 5$

$\Rightarrow \left(\frac{dS}{dt}\right)_{x=5} = -\frac{16}{5}\text{ cm}^2/\text{min}$

29 **(d)**



Let $\angle COA = \theta \Rightarrow OA = OC\sec\theta = 4\sec\theta$

Also $\angle OPC = \theta \Rightarrow OP = OC\operatorname{cosec}\theta = 4\operatorname{cosec}\theta$

Now, $\Delta_{PAB} = OA\ OP = \frac{32}{\sin 2\theta}$

For Δ_{PAB} to be minimum $\sin 2\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

$\Rightarrow P = (0, 4\sqrt{2})$

30 **(c)**

Given $y = e^{(2x^2-2x+1)\sin^2 x} = e^{2\left[\left(x-\frac{1}{2}\right)^2+\frac{1}{4}\right]\sin^2 x}$

Clearly, the minimum value occurs when

$\sin^2 x = 0$ as $\left[\left(x - \frac{1}{2}\right)^2 + \frac{1}{4}\right] \geq 1/4$

31 **(b)**

We know that there exists at least one x in (0, 1) for which

$\frac{f(1) - f(0)}{g(1) - g(0)} = \frac{f'(x)}{g'(x)}$

or $\frac{2-10}{4-2} = \frac{f'(x)}{g'(x)}$ or $f'(x) + 4g'(x) = 0$ for at least one x in(0, 1)

32 **(c)**

$f(x) = (x-1)^2 + (x-2)^2 + (x-3)^2 + (x-4)^2 + (x-5)^2$

$f'(x) = 2[x - 1 + x - 2 + x - 3 + x - 4 + x - 5] = 2[5x - 15]$

$f'(x) = 0$ gives $x = 3$ and $f''(x) > 0$ for all x

$\therefore f(x)$ is minimum for $x = 3$

33 **(c)**

$f(x) = \int e^x (x-1)(x-2)dx$

For decreasing function, $f'(x) < 0$

$\Rightarrow e^x(x-1)(x-2) < 0$

$\Rightarrow (x-1)(x-2) < 0 \Rightarrow 1 < x < 2$ $(\because e^x > 0\ \forall x \in R)$

34 **(d)**

Here, $f(x) = \log_e x$

$\therefore f'(c) = \frac{f(b) - f(a)}{b - a} \Rightarrow \frac{1}{c} = \frac{\log_e 3 - \log_e 1}{3 - 1}$

$\Rightarrow \frac{1}{c} = \frac{1}{2}\log_e 3 \Rightarrow c = 2\log_3 e$

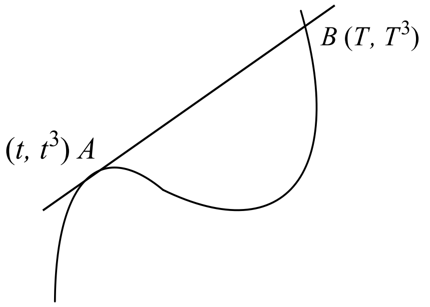
35 **(d)**

$f(x) = x^{100} + \sin x - 1$

$\Rightarrow f'(x) = 100x^{99} + \cos x$

If $0 < x < \frac{\pi}{2}$, then $f'(x) > 0$, therefore $f(x)$ is increasing on $(0, \pi/2)$
If $0 < x < 1$, then
$100x^{99} > 0$ and $\cos x > 0$ [$\because x$ lies between 0 and 1 radian]
$\Rightarrow f'(x) = 100x^{99} + \cos x > 0$
$\Rightarrow f(x)$ is increasing on (0, 1)
If $\frac{\pi}{2} < x < \pi$, then
$100x^{99} > 100$ [$\because x > 1 \Rightarrow x^{99} > 1$]
$\Rightarrow 100x^{99} + \cos x > 0$ [$\because \cos x \geq -1$
$\Rightarrow 100x^{99} + \cos x > 99$]
$\Rightarrow f'(x) > 0 \Rightarrow f(x)$ is increasing in $(\pi/2, \pi)$

36 **(a)**



$\frac{dy}{dx} = 3x^2 = 3t^2$ at A

$\therefore 3t^2 = \frac{T^3 - t^3}{T - t} = T^2 + Tt + t^2$

$\Rightarrow T^2 + Tt - 2t^2 = 0$
$\Rightarrow (T - t)(T + 2t) = 0 \Rightarrow T = t$ or $T = -2t$
($T = t$ is not possible)
Now, $m_A = 3t^2$ and $m_B = 3T^2$

$\Rightarrow \frac{m_B}{m_A} = \frac{T^2}{t^2} = \frac{4t^2}{t^2}$ (using $T = -2t$)

37 **(d)**

We have $g'(x) = f'\left(\frac{x}{2}\right) - f'(2 - x)$
Given $f''(x) < 0\ \forall x \in (0, 2)$
So, $f'(x)$ is decreasing on (0, 2)
Let $\frac{x}{2} > 2 - x \Rightarrow f'\left(\frac{x}{2}\right) < f'(2 - x)$
Thus, $\forall\ x > \frac{4}{3}, g'(x) < 0$
$\Rightarrow g(x)$ decreasing in $\left(\frac{4}{3}, 2\right)$
and increasing in $\left(0, \frac{4}{3}\right)$

38 **(b)**

Differentiating w.r.t., x, we get $1 + \frac{dy}{dx} = e^{xy}\left(y + x\frac{dy}{dx}\right)$ or

$\frac{dy}{dx} = \frac{ye^{xy} - 1}{1 - xe^{xy}}$

$\frac{dy}{dx} = \infty \Rightarrow 1 - xe^{xy} = 0$

This holds for $x = 1, y = 0$

39 **(b)**

Any point on the parabola $y^2 = 8x$ ($4a = 8$ or $a = 2$) is $(at^2, 2at)$ or $(2t^2, 4t)$
For its minimum distance from the circle means its distance from the centre $(0, -6)$ of the circle
Let D be the distance, then
$z = D^2 = (2t^2)^2 + (4t + 6)^2$
$= 4(t^4 + 4t^2 + 12t + 9)$

$\therefore \frac{dz}{dt} = 4(4t^3 + 8t + 12) = 0$

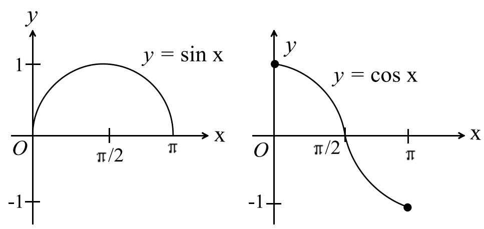
$\Rightarrow 16(t^3 + 2t + 3) = 0$
$\Rightarrow 16(t + 1)(t^2 - t + 3) = 0$
$\Rightarrow t = -1$

$\frac{d^2z}{dt^2} = 16(3t^2 + 2) = +$ve, hence minimum

$\therefore$ point is $(2, -4)$

40 **(d)**

From the graph, it is clear that both $\sin x$ and $\cos x$ in the internal $(\pi/2, \pi)$ are the decreasing functions



Therefore, S is correct
To disprove R let us consider the counter example,
$f(x) = \sin x$ in $(0, \pi/2)$
So that $f'(x) = \cos x$
Again from the graph, it is clear that $f(x)$ is increasing in $(0, \pi/2)$, but $f'(x)$ is decreasing in $(0, \pi/2)$
Therefore, R is wrong. Therefore, **d**. is the correct option

41 **(d)**

$f(x) = x\ \text{In}x - x + 1$
$\therefore f(1) = 0$
$f'(x) = 1 + \text{In}x - 1 = \text{In}x$
$\therefore f'(x) < 0$ if $0 < x < 1$
and $f'(x) > 0$ if $x > 1$

42 **(b)**

Length of sub-normal = length of sub-tangent $\Rightarrow \frac{dy}{dx} = \pm 1$

If $\frac{dy}{dx} = 1$, equation of the tangent $y - 4 = x - 3$

$\Rightarrow y - x = 1$, area of $\Delta OAB = \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$

If $\frac{dy}{dx} = -1$, equation of the tangent is

$y - 4 = -x + 3$

$\Rightarrow y + x = 7$, area $= \frac{1}{2} \times 7 \times 7 = \frac{49}{2}$

43 **(c)**

$\frac{dy}{dx} = 3x^2 - 2ax + 1$

Given that $\frac{dy}{dx} \geq 0$

$\Rightarrow 3x^2 - 2ax + 1 \geq 0$ for all x

$\Rightarrow D \leq 0$ or $4a^2 - 12 \leq 0$

$\Rightarrow -\sqrt{3} \leq a \leq \sqrt{3}$

44 **(b)**

$f\left(\frac{5\pi}{6}\right) = \log \sin\left(\frac{5\pi}{6}\right)$

$= \log \sin \frac{\pi}{6} = \log \frac{1}{2} = -\log 2,$

$f\left(\frac{\pi}{6}\right) = \log \sin \frac{\pi}{6} = -\log 2$

$f'(c) = \frac{1}{\sin x} \cos x = \cot x$

By Lagrange's mean value theorem,

$\frac{f(5\pi/6) - f(\pi/6)}{(5\pi/6) - (\pi/6)} = \cot c$

$\Rightarrow \cot c = 0 \Rightarrow c = \frac{\pi}{2}$

Thus, $c = \frac{\pi}{2} \in (\pi/6, \ 5\pi/6)$

45 **(d)**

$|f(x)| = \begin{cases} f(x), f(x) \geq 0 \\ -f(x), f(x) < 0 \end{cases}$

$\Rightarrow \frac{d}{dx}|f(x)| = \begin{cases} f'(x), f(x) > 0 \\ -f'(x), f(x) < 0 \end{cases}$

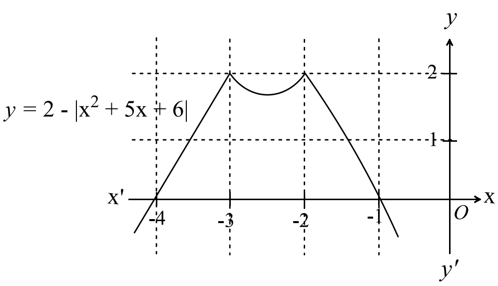
Now as $f(x)$ and $f'(x)$ keep opposite sign, then $\frac{d}{dx}|f(x)| < 0$

Hence $|f|$ is decreasing

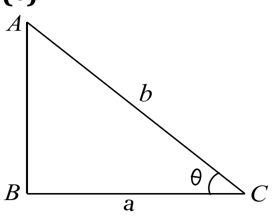
46 **(a)**

$f(x)$ will have maxima at $x = -2$ only if

$a^2 + 1 \geq 2 \Rightarrow |a| \geq 1$



47 **(c)**



$b \cos\theta = a \Rightarrow b\cos\theta + b = 4$ or $b = \frac{4}{1+\cos\theta}$

$\Rightarrow a = \frac{4\cos\theta}{1+\cos\theta}$

$\Rightarrow$ area $= \Delta = \frac{1}{2} ba \sin\theta$

$= \frac{1}{2} \frac{4}{1+\cos\theta} \frac{4\cos\theta}{1+\cos\theta} \times \sin\theta = \frac{4\sin 2\theta}{(1+\cos\theta)^2}$

$\Rightarrow \frac{d\Delta}{d\theta} = 4 \frac{2\cos 2\theta\,(1+\cos\theta)^2 + 2\sin 2\theta\,(1+\cos\theta)\sin\theta}{(1+\cos\theta)^4}$

$\Rightarrow \frac{d\Delta}{d\theta} = 0 \Rightarrow \cos 2\theta\,(1+\cos\theta) + \sin 2\theta \sin\theta = 0$

or $\cos 2\theta + \cos\theta = 0$ or $\cos 2\theta = -\cos\theta = \cos(\pi - \theta)$

or $\theta = \frac{\pi}{3}$

Therefore, Δ is maximum when $\theta = \frac{\pi}{3}$

48 **(d)**

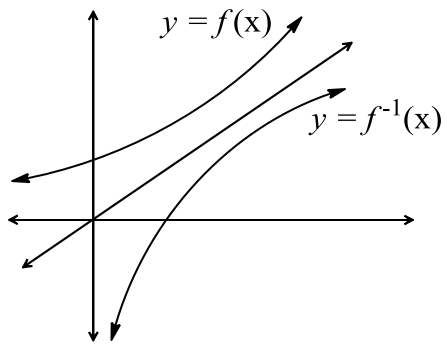
$x = 2 \text{ In} \cot t + 1, y = \tan t + \cot t$

Slope of the tangent

$\left(\frac{dy}{dx}\right)_{t=\frac{\pi}{4}} = \left(\frac{\sec^2 t - \text{cosec}^2 t}{-\frac{2}{\cot t}\text{cosec}^2 t}\right)_{t=\frac{\pi}{4}} = 0$

49 **(d)**

If $f(x)$ increases then $f^{-1}(x)$ increases. Refer figure



If $f(x)$ increases, then $f'(x) > 0$

$\Rightarrow \frac{d}{dx}\left(\frac{1}{f(x)}\right) = -\frac{f'(x)}{f^2(x)} < 0 \Rightarrow \frac{1}{f(x)}$ decreases

$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}$

If f and g are +ve functions and $f' < 0$ and $g' < 0$, then $\frac{d}{dx}\left(\frac{f}{g}\right) < 0$

50 **(d)**

$f'(x) = a + 3\cos x - 4\sin x$

$= a + 5\cos(x+\alpha)$, where $\cos\alpha = \frac{3}{5}$

For invertible, $f(x)$ must be monotonic

$\Rightarrow f'(x) \geq 0\ \forall x$ or $f'(x) \leq 0\ \forall x$

$\Rightarrow a + 5\cos(x+\alpha) \geq 0$ or $a + 5\cos(x+\alpha) \leq 0$

$\Rightarrow a \geq -5\cos(x+\alpha)$ or $a \leq -5\cos(x+\alpha)$

$\Rightarrow a \geq 5$ or $a \leq -5$

51 **(d)**

Differentiating $y^3 - x^2y + 5y - 2x = 0$ w. r.t. x, we get

$3y^2y' - 2xy - x^2y' + 5y' - 2 = 0$

$\Rightarrow y' = \dfrac{2xy+2}{3y^2 - x^2 + 5} \Rightarrow y'_{(0,0)} = 2/5$

Differentiating $x^4 - x^3y^2 + 5x + 2y = 0$ w.r.t. x,

We have $4x^3 - 3x^2y^2 - 2x^3yy' + 5 + 2y' = 0$

$\Rightarrow y' = \dfrac{3x^2y^2 - 4x^3 - 5}{2 - 2x^3y} \Rightarrow y'_{(0,0)} = -5/2$

Thus, both the curves intersect at right angle

52 **(d)**

Let $f(x) = x^3 + 2x^2 + 5x + 2\cos x$

$\Rightarrow f'(x) = 3x^2 + 4x + 5 - 2\sin x$

Now the least value of $3x^2 + 4x + 5$ is

$-\dfrac{D}{4a} = -\dfrac{(4)^2 - 4(3)(5)}{4(3)} = \dfrac{11}{3}$

And the greatest value of $2\sin x = 2$

$\Rightarrow 3x^2 + 4x + 5 > 2\sin x \ \forall x \in R$

$\Rightarrow f'(x) = 3x^2 + 4x + 5 - 2\sin x > 0 \ \forall x \in R$

$\Rightarrow f(x)$ is strictly an increasing function also

$f(0) = 2$ and $f(2\pi) > 0$

Thus, for the given interval, $f(x)$ never becomes zero

Hence, the number of roots is zero

53 **(c)**

$x^2y = c^3$

Differentiating w.r.t. x, we have

$x^2\dfrac{dy}{dx} + 2xy = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{2y}{x}$

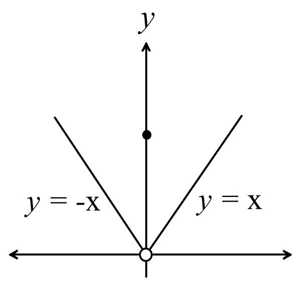
Equation of the tangent at (h,k) $y - k = -\frac{2k}{h}(x - h)$

$y = 0$ gives $x = \frac{3h}{2} = a$, and $x = 0$ gives

$y = 3k = b$

Now, $a^2b = \frac{9h^2}{4}3k = \frac{27}{4}h^2k = \frac{27}{4}c^3$

54 **(a)**



From the graph $f(0^+) < f(0)$ and $f(0^-) < 0 \Rightarrow$ $x = 0$ is the point of maxima

55 **(c)**

Consider the function $f(x) = \frac{x^2}{(x^3+200)}$ (1)

$f'(x) = x\dfrac{(400 - x^3)}{(x^3 + 200)^2} = 0$

When $x = (400)^{1/3}$, $(\because x \neq 0)$

$x = (400)^{1/3} - h \Rightarrow f'(x) > 0$

$x = (400)^{1/3} + h \Rightarrow f'(x) < 0$

$\therefore f(x)$ has maxima at $x = (400)^{1/3}$

Since $7 < (400)^{1/3} < 8$, either a_7 or a_8 is the greatest term of the sequence

$\because a_7 = \frac{49}{543}$ and $a_8 = \frac{8}{89}$ and $\frac{49}{543} > \frac{8}{89}$

$\Rightarrow a_7 = \frac{49}{543}$ is the greatest term

56 **(a)**

Let the required point be (x_1, y_1)

Now, $3y = 6x - 5x^3$

$\Rightarrow 3\dfrac{dy}{dx} = 6 - 15x^2$

$\Rightarrow \dfrac{dy}{dx} = 2 - 5x^2$

$\Rightarrow \left(\dfrac{dy}{dx}\right)_{(x_1,y_1)} = 2 - 5x_1^2$

The equation of the normal at (x_1, y_1) is

$y - y_1 = \dfrac{-1}{2 - 5x_1^2}(x - x_1)$

If it passes through the origin, then

$0 - y_1 = \dfrac{-1}{2 - 5x_1^2}(0 - x_1)$

$\Rightarrow y_1 = \frac{-x_1}{2-5x_1^2}$ (1)

Since (x_1, y_1) lies on the given curve

Therefore, $3y_1 = 6x_1 - 5x_1^3$ (2)

Solving equations (1) and (2), we obtain $x_1 = 1$ and $y_1 = 1/3$

Hence, the required point is $(1, 1/3)$

57 **(c)**

$u = \sqrt{c+1} - \sqrt{c}$

$u = \frac{1}{\sqrt{c+1}+\sqrt{c}}$ and $v = \frac{1}{\sqrt{c-1}+\sqrt{c}}$

Clearly $u < v$

Also, f is increasing whereas g is decreasing

Thus $u < v$

$\Rightarrow f(u) < f(v)$

$\Rightarrow gof(u) > gof(v)$

58 **(d)**

$g'(x) = \left(f'((\tan x - 1)^2 + 3)\right)2(\tan x - 1)\sec^2 x$

Since $f''(x) > 0 \Rightarrow f'(x)$ is increasing

So, $f'((\tan x - 1)^2 + 3) > f'(3) = 0 \ \forall x \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

Also, $(\tan x - 1) > 0$ $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

So, $g(x)$ is increasing in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

59 **(c)**

$f'(x) = 3kx^2 - 18x + 9 = 3[kx^2 - 6x + 3]$
$\geq 0, \forall x \in R$

$\Rightarrow D = b^2 - 4ac \leq 0, k > 0$, i.e., $36 - 12k \leq 0$

$\Rightarrow k \geq 3$

60 **(b)**

Since $\cos\theta \le 1$ for all θ. Therefore, $f(x) \le 1$ for all x

61 **(a)**

Given $\phi'(x) - \phi(x) > 0\ \forall x \ge 1$

$\Rightarrow e^{-x}\{\phi'(x) - \phi(x)\} > 0\ \forall x \ge 1$

$\Rightarrow \frac{d}{dx} e^{-x}\phi(x) > 0\ \forall x \ge 1$

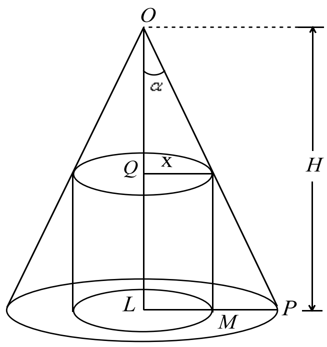
$\therefore e^{-x}\phi(x)$ is an increasing function $\forall x \ge 1$

Since $\phi(x)$ is a polynomial

$\Rightarrow e^{-x}\phi(x) > e^{-1}\phi(1) \Rightarrow e^{-x}\phi(x) > 0 \quad [\because \phi(1) = 0]$

$\Rightarrow \phi(x) > 0$

62 **(a)**



Let H be the height of the cone and α be its semi-vertical angle. Suppose that x is the radius of the inscribed cylinder and h be its height

$h = QL = OL - OQ = H - x\cot\alpha$, $V =$ volume of the cylinder $= \pi x^2(H - x\cot\alpha)$

Also, $p = \frac{1}{3}\pi(H\tan\alpha)^2 H \quad (1)$

$\frac{dV}{dx} = \pi(2Hx - 3x^2\cot\alpha)$

So, $\frac{dV}{dx} = 0 \ \Rightarrow x = 0$ or $x = \frac{2}{3}H\tan\alpha$;

$\left.\frac{d^2V}{dx^2}\right|_{x=\frac{2}{3}H\tan\alpha} = -2\pi H < 0$

So, V is maximum when $x = \frac{2}{3}H\tan\alpha$

$q = V_{\max} = \pi\frac{4}{9}H^2\tan^2\alpha\frac{1}{3}H$

$= \frac{4}{27}\frac{\pi^3 p\tan^2\alpha}{\pi\tan^2\alpha} = \frac{4}{9}p \quad$ [from (1)]

Hence, $p : q = 9 : 4$

63 **(a)**

Here $y > 0$. Putting $y = x$ in $y = \sqrt{4 - x^2}$, we get $x = \sqrt{2}, -\sqrt{2}$

So, the point is $(\sqrt{2}, \sqrt{2})$

Differentiating $y^2 + x^2 = 4$ w.r.t. x,

$2y\frac{dy}{dx} + 2x = 0$ or $\frac{dy}{dx} = -\frac{x}{y}$

$\Rightarrow$ at $(\sqrt{2}, \sqrt{2})$, $\frac{dy}{dx} = -1$

64 **(b)**

$f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$

$\therefore f'(x) = 6x^2 - 18ax + 12a^2$ and $f''(x) = 12x - 18a$

For maximum/minimum, $6x^2 - 18ax + 12a^2 = 0$

$\Rightarrow x^2 - 3ax + 2a^2 = 0$

$\Rightarrow (x - a)(x - 2a) = 0$

$\Rightarrow x = a$ or $x = 2a$

Now, $f''(a) = 12a - 18a = -6a < 0$

and $f''(2a) = 24a - 18a = 6a > 0$

$\therefore f(x)$ is maximum at $x = a$ and minimum at $x = 2a$

$\Rightarrow p = a$ and $q = 2a$

Given that $p^2 = q \Rightarrow a^2 = 2a \Rightarrow a(a-2) = 0 \Rightarrow a = 2$

65 **(c)**

When $f''(a) = 0$, then $f'''(a)$ must also be zero and sign of $f''''(a)$ will decide about maximum or minimum

66 **(b)**

Let $f(x) = \frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx$,

Which is continuous and differentiable

$f(0) = 0, f(-1) = \frac{a}{4} - \frac{b}{3} + \frac{c}{2} - d$

$= \frac{1}{4}(a + 2c) - \frac{1}{3}(b + 3d) = 0$

So, according to Rolle's theorem, there exists at least one root of $f'(x) = 0$ in $(-1, 0)$

67 **(a)**

Since $a = \left(\frac{4}{\sin x} + \frac{1}{1 - \sin x}\right)$, a is least

$\Rightarrow \frac{da}{dx} = \left[-\frac{4}{\sin^2 x} + \frac{1}{(1 - \sin x)^2}\right]\cos x = 0$

We have to find the values of x in the interval $(0, \pi/2)$

$\Rightarrow \cos x \ne 0$ and the other factor when equated to zero gives $\sin x = 2/3$

Now, $\frac{d^2a}{dx^2} = \left[-\frac{4}{\sin^2 x} + \frac{1}{(1-\sin x)^2}\right](-\sin x) + \left[\frac{8}{\sin^3 x} + \frac{2}{(1-\sin x)^3}\right]\cos^2 x$

Put $\sin x = \frac{2}{3}$ and $\cos^2 x = 1 - \frac{4}{9} = \frac{5}{9}$

$\therefore \frac{d^2a}{dx^2} = 0 + \left[\frac{8}{8/27} + 2 \times 27\right]\frac{5}{9} = 81 \times \frac{5}{9} = 45 > 0$

$\Rightarrow a$ is minimum and its value is

$\frac{4}{2/3} + \frac{1}{1 - (2/3)} = 6 + 3 = 9$

68 **(d)**

Given, $V = \pi r^2 h$

Differentiating both sides, we get

$\frac{dV}{dt} = \pi\left(\frac{r^2 dh}{dt} + 2r\frac{dr}{dt}h\right) = \pi r\left(r\frac{dh}{dt} + 2h\frac{dr}{dt}\right)$

$\frac{dr}{dt} = \frac{1}{10}$ and $\frac{dh}{dt} = -\frac{2}{10}$

$\frac{dV}{dt} = \pi r\left(r\left(-\frac{2}{10}\right) + 2h\left(\frac{1}{10}\right)\right) = \frac{\pi r}{5}(-r + h)$

Thus, when $r = 2$ and $h = 3$,

$\frac{dV}{dt} = \frac{\pi(2)}{5}(-2+3) = \frac{2\pi}{5}$

69 **(b)**

$$f'(x) = \frac{x \text{ In}\left(\frac{e+x}{\pi+x}\right) + \left(e \text{ In}(e+x) - \pi \text{In}(\pi+x)\right)}{\left(\text{In}(e+x)\right)^2(\pi+x)(e+x)}$$

Now $\pi + x > e + x \Rightarrow \text{In}\left(\frac{e+x}{\pi+x}\right) < 0 \Rightarrow f'(x) < 0$

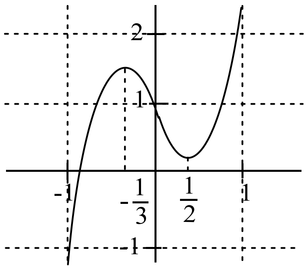
and also $e \text{ In}(e+x) < \pi \text{ In}(\pi+x) \Rightarrow f'(x) < 0$

Thus, $f(x)$ is decreasing

70 **(d)**

$f'(x) = 12x^2 - 2x - 2 = 2[6x^2 - x - 1]$

$= 2(3x+1)(2x-1)$



Hence $g(x) = \begin{cases} f(x), \text{if } 0 \le x < \frac{1}{2} \\ f\left(\frac{1}{2}\right), \text{if } \frac{1}{2} \le x \le 1 \\ 3 - x, \text{if } 1 < x \le 2 \end{cases}$

$\Rightarrow g\left(\frac{1}{4}\right) + g\left(\frac{3}{4}\right) + g\left(\frac{5}{4}\right) = f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) + g\left(\frac{5}{4}\right)$

$= \frac{5}{2}$

71 **(b)**

Applying LMVT in [0, 1] to the function $y = f(x)$, we get

$f'(c) = \frac{f(1) - f(0)}{1-0}$, for some $c \in (0,1)$

$\Rightarrow e^{c^2} = \frac{f(1) - f(0)}{1}$

$\Rightarrow f(1) - 10 = e^{c^2}$ for some $c \in (0,1)$

But $1 < e^{c^2} < e$ in $(0,1)$

$\Rightarrow 1 < f(1) - 10 < e$

$\Rightarrow 11 < f(1) < 10 + e$

$\Rightarrow A = 11, B = 10 + e$

$\Rightarrow A - B = 1 - e$

72 **(d)**

Using Lagrange's mean value theorem, for some $c \in (1,6)$

such that $f'(c) = \frac{f(6) - f(1)}{5} = \frac{f(6)+2}{5} \ge 4.2$

$\Rightarrow f(6) + 2 \ge 21$

$\Rightarrow f(6) \ge 19$

73 **(c)**

The equation of the line is $y - 3 = \frac{3+2}{0-5}(x - 0)$, i.e.,

$x + y - 3 = 0$

$y = \frac{c}{x+1} \Rightarrow \frac{dy}{dx} = \frac{-c}{(x+1)^2}$

Let the line touches the curve at (α, β)

$\Rightarrow \alpha + \beta - 3 = 0, \left(\frac{dy}{dx}\right)_{\alpha,\beta} = \frac{-c}{(\alpha+1)^2} = -1$ and

$\beta = \frac{c}{\alpha + 1}$

$\Rightarrow \frac{c}{(c/\beta)^2} = 1$ or $\beta^2 = c$ or $(3-\alpha)^2 = c$

$= (\alpha+1)^2$

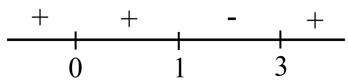
$\Rightarrow 3 - \alpha = \pm(\alpha + 1)$ or $3 - \alpha = \alpha + 1$

$\Rightarrow \alpha = 1$. So, $c = (1+1)^2 = 4$

74 **(b)**

$\frac{dy}{dx} = 5x^2(x-1)(x-3) = 0$

$\therefore x = 0, 1, 3$



Clearly $x = 0$ is neither a point of maxima nor a point of minima as derivative does not change sign at $x = 0$

$x = 1$ is a point of maxima and $x = 3$ is a point of minima

75 **(b)**

$y = -x^3 + 3x^2 + 9x - 27$

$\therefore \frac{dy}{dx} = -3x^2 + 6x + 9$

Let the slope of tangent to the curve at any point be m (say)

$\Rightarrow m = -3x^2 + 6x + 9 \Rightarrow \frac{dm}{dx} = -6x + 6$

$\frac{d^2m}{dx^2} = -6 < 0$ for all x

Therefore, m is maximum when $\frac{dm}{dx} = 0$, i.e., when $x = 1$

Therefore, maximum slope $= -3 + 6 + 9 = 12$

76 **(d)**

$f(x) = (1+b^2)x^2 + 2bx + 1$

The graph of $f(x)$ is upward parabola as coefficient of x^2 is $1 + b^2 > 0$

$\Rightarrow$ The range of $f(x)$ is $\left[\frac{-D}{4a}, \infty\right)$, where D is discriminant of $f(x)$

$\Rightarrow m(b) = -\frac{4b^2 - 4(1+b^2)}{4(1+b^2)}$

$\Rightarrow m(b) = \frac{1}{1+b^2} \in (0, 1]$

77 **(a)**

$y^2 = \alpha x^3 - \beta \Rightarrow \frac{dy}{dx} = \frac{3\alpha x^2}{2y}$

$\Rightarrow$ Slope of the normal at (2, 3) is

$\left(-\frac{dx}{dy}\right)_{(2,3)} = -\frac{2 \times 3}{3\,\alpha(2)^2} = -\frac{1}{2\alpha} = -\frac{1}{4}$

$\Rightarrow \alpha = 2$

Also, (2, 3) lies on the curve

$\Rightarrow 9 = 8\alpha - \beta \Rightarrow \beta = 16 - 9 = 7 \Rightarrow \alpha + \beta = 9$

78 **(c)**

$f'(x) = \frac{(1 + 4x + x^2)1 - x(4 + 2x)}{(1 + 4x + x^2)^2}$

$= \frac{1 - x^2}{(1 + 4x + x^2)^2}$

For maximum or minimum $f'(x) = 0 \Rightarrow x = \pm 1$

For $x = 1, f'(x)$ changes sign from positive to negative as x passes through 1

Therefore, $f(x)$ is maximum for $x = 1$, and maximum value $= \frac{1}{1+4+1} = \frac{1}{6}$

79 **(a)**

Here $f(x) = \tan^{-1}(\sin x + \cos x)$

$\therefore f'(x) = \frac{1}{1 + (\sin x + \cos x)^2}(\cos x - \sin x)$

$= \frac{\cos x - \sin x}{2 + \sin 2x}$

For $-\frac{\pi}{2} < x < \frac{\pi}{4}, \cos x > \sin x$

Hence, $y = f(x)$ is increasing in $\left(-\frac{\pi}{2}, \frac{\pi}{4}\right)$

80 **(c)**

Solving $y = |x^2 - 1|$ and $y = \sqrt{7 - x^2}$

We have $|x^2 - 1| = \sqrt{7 - x^2}$

$\Rightarrow x^4 - 2x^2 + 1 = 7 - x^2$

$\Rightarrow x^4 - x^2 - 6 = 0$

$\Rightarrow (x^2 - 3)(x^2 + 2) = 0$

$\Rightarrow x = \pm\sqrt{3}$

Points of intersection of the curves $y = |x^2 - 1|$ and $y = \sqrt{7 - x^2}$ are $(\pm\sqrt{3}, 2)$

Since both the curves are symmetrical about the y-axis, points of intersection are also symmetrical

Now, $y = x^2 - 1 \Rightarrow \frac{dy}{dx} = 2x$

$\Rightarrow m_1 = \left.\frac{dy}{dx}\right|_{(\sqrt{3},2)} = 2\sqrt{3}$

And $y = \sqrt{7 - x^2} \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$

$\Rightarrow m_2 = \left.\frac{dy}{dx}\right|_{(\sqrt{3},2)} = -\frac{\sqrt{3}}{2} \Rightarrow \tan\theta = \left|\frac{5\sqrt{3}}{4}\right|$

81 **(c)**

$f(f(x)) = k(x^5 + x) \Rightarrow f'(f(x))f'(x)$

$= k(5x^4 + 1)$

$\Rightarrow f(x)$ is always increasing or decreasing as $k(5x^4 + 1)$ is either always negative or positive

82 **(b)**

$f(x) = |x \log_e x|$

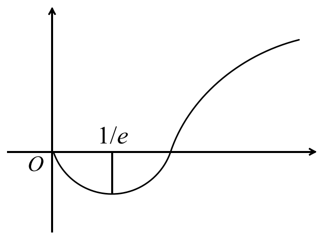
For $g(x) = x \log_e x$,

$g'(x) = x\frac{1}{x} + \log_e x = 1 + \log_e x$

$\Rightarrow g(x)$ increases for $\left(\frac{1}{e}, \infty\right)$ and decreases for $\left(0, \frac{1}{e}\right)$

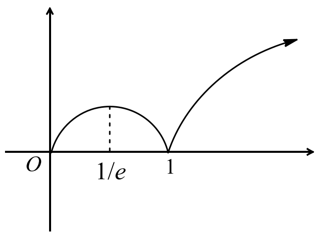
Graph of $y = g(x) = x \log_e x$

(a)



Graph of $y = f(x) = |x \log_e x|$

(b)



From the graph, $f(x) = |x \log_e x|$ decreases in $\left(\frac{1}{e}, 1\right)$

83 **(d)**

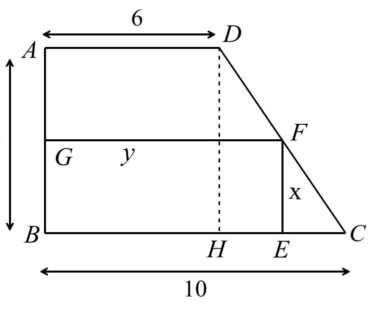
$f(0) > f(0^+)$ and $f(0) < f(0^-)$, hence $x = 0$ is neither a maximum nor a minimum

84 **(d)**

Let there be a value of k for which $x^3 - 3x + k = 0$ has two distinct roots between 0 and 1

Let a, b be two distinct roots of $x^3 - 3x + k = 0$ lying between 0 and 1 such that $a < b$. Let $f(x) = x^3 - 3x + k$. Then $f(a) = f(b) = 0$. Since between any two roots of a polynomial $f(x)$, there exists at least one root of its derivative $f'(x)$. Therefore, $f'(x) = 3x^2 - 3$ has at least one root between a and b. But $f'(x) = 0$ has two roots equal to ± 1 which do not lie between a and b. Hence $f(x) = 0$ has no real roots lying between 0 and 1 for any value of k

85 **(b)**



Let rectangle $BEFG$ is inscribed
Its area, $A = xy$
Now ΔFEC and ΔDHC are similar, i.e.,
$\Rightarrow \frac{x}{8} = \frac{10-y}{4} \Rightarrow y = 10 - \frac{x}{2} \Rightarrow A = x\left(10 - \frac{x}{2}\right)$
where $x \in (0, 8]$
Now $\frac{dA}{dx} = 10 - x$. Now for $x \in (0, 8)$
$\frac{dA}{dx} > 0 \Rightarrow A$ increases. Hence $A_{\max}$ occurs when $x = 8$
Hence, max area $= A_{\max} = 8\left(10 - \frac{8}{2}\right) = 48\text{ cm}^2$

86 **(d)**
$f(x)$ vanishes at points where
$\sin\frac{\pi}{x} = 0$, i.e., $\frac{\pi}{x} = k\pi, k = 1, 2, 3, 4, \ldots$
Hence $x = \frac{1}{k}$
Also $f'(x) = \sin\frac{\pi}{x} - \frac{\pi}{x}\cos\frac{\pi}{x}$, if $x \neq 0$
Since the function has a derivative at any interior point of the interval (0, 1), also continuous in [0, 1] and $f(0) = f(1)$. Hence, Rolle's theorem is applicable to any one of the interval
$\left[\frac{1}{2}, 1\right], \left[\frac{1}{3}, \frac{1}{2}\right], \ldots, \left[\frac{1}{k+1}, \frac{1}{k}\right]$
Hence, there exists at least one c in each of these intervals where $f'(c) = 0 \Rightarrow$ infinite points

87 **(a)**
$$f'(x) = -\frac{1}{2}e^{-\frac{x}{2}}(x^2 - 8)$$
Clearly, $x = 2\sqrt{2}$ is the point of local maxima

88 **(b)**
$f(x) = (x-8)^4(x-9)^5, 0 \leq x \leq 10$
$\Rightarrow f'(x) = 4(x-8)^3(x-9)^5 + 5(x-9)^4(x-8)^4$
$= (x-8)^3(x-9)^4[4(x-9) + 5(x-8)]$
$= 9(x-8)^3(x-9)^4\left(x - \frac{76}{9}\right)$
Sign scheme of $f'(x)$

+   -   +   +
8   76/9   9

$f'(x) < 0$, if $x \in \left(8, \frac{76}{9}\right) \Rightarrow f(x)$ decreases if $x \in \left(8, \frac{76}{9}\right)$

89 **(c)**
$\frac{a}{x^2} + \frac{b}{y^2} = 1 \Rightarrow ay^2 + bx^2 = x^2y^2$ (1)
$$-\frac{2a}{x^3} - \frac{2b}{y^3}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{ay^3}{bx^3}$$
Equation of the tangent at (h, k) is $y - k = -\frac{ak^3}{bh^3}(x - h)$
For x-intercept, put $y = 0$
$$\Rightarrow x = \frac{bh^3}{ak^2} + h \Rightarrow x = h\left[\frac{bh^2 + ak^2}{ak^2}\right] = h\left[\frac{h^2k^2}{ak^2}\right]$$
$$= \frac{x^3}{a}$$
$\Rightarrow$ x-intercept is proportional to the cube of abscissa

90 **(a)**
$$f(x) = \frac{\sin^3 x \cos x}{2}$$
$$\Rightarrow f'(x) = \frac{3\sin^2 x\cos^2 x - \sin^4 x}{2}$$
$f'(x) = 0 \Rightarrow 3\sin^2 x\cos^2 x - \sin^4 x = 0$
$\Rightarrow 3\cos^2 x - \sin^2 x = 0$
$\Rightarrow 4\cos^2 x - 1 = 0$
$\Rightarrow \cos x = \frac{1}{2}$
$\Rightarrow x = \frac{\pi}{3}$, which is the point of maxima
$$\Rightarrow f_{\max} = \frac{\left(\sqrt{3}/2\right)^3(1/2)}{2} = \frac{3\sqrt{3}}{32}$$

91 **(b)**
Equation of tangent at $(3\sqrt{3}, \cos\theta, \sin\theta)$ is
$$\frac{x\cos\theta}{3\sqrt{3}} + \frac{y\sin\theta}{1} = 1$$
Thus, sum of intercepts$= \left(3\sqrt{3}\sec\theta + \operatorname{cosec}\theta\right) = f(\theta)$ [say]
$$\Rightarrow \quad f'(\theta) = \frac{3\sqrt{3}\sin^3\theta - \cos^3\theta}{\sin^2\theta\cos^2\theta}$$
Put $f'(\theta) = 0$
$$\therefore \quad \sin^3\theta = \frac{1}{3^{3/2}}\cos^3\theta$$
$$\Rightarrow \quad \tan\theta = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad \theta = \frac{\pi}{6}$$
Also, for $0 < \theta < \frac{\pi}{6}, \frac{dz}{d\theta} < 0$ and for $\frac{\pi}{6} < \theta < \frac{\pi}{2}, \frac{dz}{d\theta} > 0$
$\therefore$Minimum at $\theta = \frac{\pi}{6}$

92 **(d)**

To satisfy Rolle's theorem, it should be continuous in [0, 1].

$\Rightarrow \lim_{n\to 0^+} f(x) = f(0)$

$\Rightarrow \lim_{n\to 0^+} \frac{\log x}{x^{-a}} = 0$

$\Rightarrow \lim_{n\to 0^+} \frac{1/x}{-\alpha x^{-a-1}} = 0$ [using L'Hospital's rule]

$\Rightarrow \lim_{n\to 0^+} -\frac{1}{\alpha x^{-\alpha}} = 0$

$\Rightarrow \lim_{n\to 0^+} -\frac{1}{\alpha} x^{\alpha} = 0$

Which shows $\alpha > 0$otherwise, it would be discontinuous also when

$\alpha > 0, f(x)$ is differentiable in $(0,1) and\ f(1) = f(0) = 0.$

Clearly $\alpha > 0$, thus $\alpha = \frac{1}{2}$is the possible answer.

93 **(d)**

Slope of the tangent $y = f(x)$ is $\frac{dy}{dx} = f'(x)_{(3,4)}$

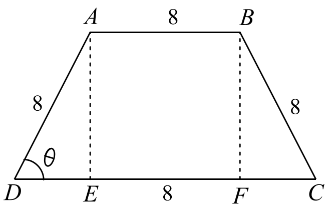
Therefore, slope of the normal $= -\frac{1}{f'(x)_{(3,4)}}$

$= -\frac{1}{f'(3)}$

$= \tan\left(\frac{3\pi}{4}\right)$ (given)

$\Rightarrow f'(3) = 1$

94 **(b)**



$\Delta = (AB \times AE) + 2\left(\frac{1}{2} DE \times AE\right)$

$= (8 \times 8 \sin\theta) + 8\sin\theta \times 8\cos\theta$

$= 64\sin\theta + 32\sin 2\theta$

Let $\frac{d\Delta}{d\theta} = 0 \Rightarrow 64\cos\theta + 64\cos 2\theta = 0$

$\Rightarrow 2\cos^2\theta + \cos\theta - 1 = 0$

$\Rightarrow (2\cos\theta - 1)(\cos\theta + 1) = 0$

$\Rightarrow \cos\theta = 1/2 \Rightarrow \theta = \pi/3$

$\Rightarrow A_{\max} = 64\frac{\sqrt{3}}{2} + 32\frac{\sqrt{3}}{2} = 32\sqrt{3} + 16\sqrt{3}$

$= 48\sqrt{3}$

95 **(b)**

$f'(x) = -\pi\sin\pi x + 10 + 6x + 3x^2$

$= 3(x+1)^2 + 7 - \pi\sin\pi x > 0$ for all x

$\therefore f(x)$ is increasing in $-2 \le x \le 3$

So, the absolute minimum $= f(-2) = 1 - 20 + 12 - 8$

96 **(d)**

$f(x) = ax^3 + bx^2 + 11x - 6$

Satisfies conditions of Rolle's theorem in [1, 3]

$\Rightarrow f(1) = f(3)$

$\Rightarrow a + b + 11 - 6 = 27a + 9b + 33 - 6$

$\Rightarrow 13a + 4b = -11$ (1)

and $f'(x) = 3ax^2 + 2bx + 11$

$\Rightarrow f'\left(2 + \frac{1}{\sqrt{3}}\right) = 3a\left(2 + \frac{1}{\sqrt{3}}\right)^2 + 2b\left(2 + \frac{1}{\sqrt{3}}\right) + 11 = 0$

$\Rightarrow 3a\left(4 + \frac{1}{3} + \frac{4}{\sqrt{3}}\right) + 2b\left(2 + \frac{1}{\sqrt{3}}\right) + 11 = 0$ (2)

From equations (1) and (2), we get $a = 1, b = -6$

97 **(a)**

Since the same line is tangent at one point $x = a$ and normal at other point $x = b$

$\Rightarrow$ Tangent at $x = b$ will be perpendicular to tangent at $x = a$

$\Rightarrow$ Slope of tangent changes from positive to negative or negative to positive. Therefore, it takes the value zero somewhere. Thus, there exists a point $c \in (a, b)$ where $f'(c) = 0$

98 **(c)**

We have $f(x) = \begin{cases} (-1)^{m+n} x^m (x-1)^n, \text{if } x < 0 \\ (-1)^n x^m (x-1)^n, \text{if } 0 \le x < 1 \\ x^m (x-1)^n, \quad \text{if } x \ge 1 \end{cases}$

Let $g(x) = x^m(x-1)^n$, then

$g'(x) = mx^{m-1}(x-1)^n + nx^m(x-1)^{n-1}$

$= x^{m-1}(x-1)^{n-1}\{mx - m + nx\}$

Now $f'(x) = 0 \Rightarrow g'(x) = 0 \Rightarrow x = 0, 1$ or $\frac{m}{m+n}$

$f(0) = 0, f(1) = 0$ and

$f\left(\frac{m}{m+n}\right) = (-1)^n \frac{m^m n^n (-1)^n}{(m+n)^{m+n}}$

$= \frac{m^m n^n}{(m+n)^{m+n}} > 0$

$\therefore$ the maximum value $= \frac{m^m n^n}{(m+n)^{m+n}}$

99 **(c)**

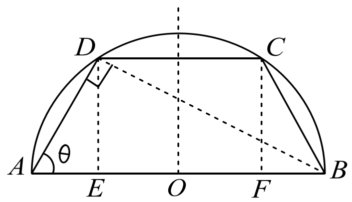
$f(x)f'(x) < 0\ \forall x \in R$

$\Rightarrow \frac{1}{2}\frac{d}{dx}\left(f^2(x)\right) < 0$

$\Rightarrow \frac{d}{dx}\left(f^2(x)\right) < 0$

$\Rightarrow f^2(x)$ is a decreasing function

100 **(a)**



$AD = AB\cos\theta = 2R\cos\theta$, $AE = AD\cos\theta$

$= 2R\cos^2\theta$

$\Rightarrow EF = AB - 2AE = 2R - 4R\cos^2\theta$

$DE = AD\sin\theta = 2R\sin\theta\cos\theta$

$\Rightarrow$ Area of trapezium,

$S = \dfrac{1}{2}(AB + CD) \times DE$

$= \dfrac{1}{2}(2R + 2R - 4R\cos^2\theta) \times 2R\sin\theta\cos\theta$

$= 4R^2\sin^3\theta\cos\theta$

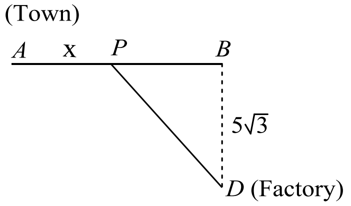
$\dfrac{dS}{d\theta} = 12R^2\sin^2\theta\cos^2\theta - 4R^2\sin^4\theta$

$= 4R^2\sin^2\theta\,(3\cos^2\theta - \sin^2\theta)$

For maximum area, $\frac{dS}{d\theta} = 0 \Rightarrow \tan^2\theta = 3 \Rightarrow \tan\theta = \sqrt{3}$

(θ is acute) $\Rightarrow S_{\max} = \frac{3\sqrt{3}}{4}R^2$

101 **(a)**



Let the charges for railway line be k Rs/km

Now the total for eight charges, $T = kx + 2k\sqrt{(20-x)^2 + 75}$

Let $\frac{dT}{dx} = 0 \Rightarrow k + 2k\,\frac{2(x-20)}{2\sqrt{(x-20)^2+75}} = 0$

$\Rightarrow 4(x-20)^2 = 75 + (x-20)^2$

$\Rightarrow (x-20)^2 = 25 \Rightarrow x = 25, 15 \Rightarrow x = 15$ (as $AP < AB$)

$\Rightarrow PB = AB - AP = 20 - 15 = 5$ km

102 **(d)**

Consider a function $g(x) = xf(x)$

Since $f(x)$ is continuous, $g(x)$ is also continuous in [0, 1] and differentiable in (0, 1)

As $f(1) = 0$

$\therefore$ $g(0) = 0 = g(1)$

Hence Rolle's theorem is applicable for $g(x)$

Therefore, there exists at least one $c \in (0,1)$ such that $g'(c) = 0$

$\Rightarrow xf'(x) + f(x) = 0$

$\Rightarrow cf'(c) + f(c) = 0$

103 **(a)**

We have $f(x) = \frac{x}{2} + \frac{2}{x}$

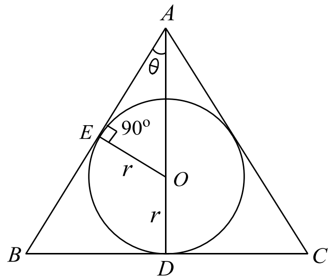
$\therefore f'(x) = \frac{1}{2} - \frac{2}{x^2}$ and $f''(x) = \frac{4}{x^3}$

Now $f'(x) = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

$\therefore f''(x) > 0$ for $x = 2$

Therefore, f has local minima at $x = 2$

104 **(c)**



Let ABC be an isosceles triangle in which a circle of radius r is inscribed

Let $\angle BAD = \theta$ (semi-vertical angle)

In $\Delta OAE, OA = OE\operatorname{cosec}\theta = r\operatorname{cosec}\theta, AE = r\cot\theta$

$\Rightarrow AD = OA + OD = r(\operatorname{cosec}\theta + 1)$

In $\Delta ABD, BD = AD\tan\theta = r(\operatorname{cosec}\theta + 1)\tan\theta$

$AB = AD\sec\theta = r(\operatorname{cosec}\theta + 1)\sec\theta$

Now, the perimeter of the ΔABC is $S = AB + AC + BC$

$= 2AB + 2BD \quad (\because AC = AB)$

$S = 2r(\operatorname{cosec}\theta + 1)(\sec\theta + \tan\theta)$ or

$S = \frac{4r(1+\sin\theta)^2}{\sin 2\theta}$

$\Rightarrow \dfrac{dS}{d\theta} = 4r\,[2(1+\sin\theta)\cos\theta\sin 2\theta - (1+\sin\theta)^2 2\cos 2\theta]/(\sin 2\theta)^2$

$= 8r(1+\sin\theta)[\sin 2\theta\cos\theta - \cos 2\theta\sin\theta - \cos 2\theta]/(\sin 2\theta)^2$

$= 8r(1+\sin\theta)(\sin\theta - 1 + 2\sin^2\theta)/(\sin 2\theta)^2$

$= 16r(1+\sin\theta)^2(\sin\theta - 1/2)/(\sin 2\theta)^2$

For maximum or minimum of $S, dS/d\theta = 0 \Rightarrow \sin\theta = 1/2$

$\therefore \theta = \pi/6$ $\quad (\because \sin\theta \neq -1$ as θ is an acute angle)

Now if θ is little less and little greater than $\pi/6$, then sign of $dS/d\theta$ changes from –ve to +ve.

Hence S is minimum when $\theta = \pi/6$, which is the point of minima

Hence, the least perimeter of the

$\Delta = 4r[1 + \sin(\pi/6)]^2/\sin(\pi/3) = 6\sqrt{3}r$

105 **(a)**

$f(0) = \pi/2, f(0^+), f(0^-) = 0$

Hence $x = 0$ is the point of maxima

106 **(b)**

Let $y = x^{25}(1-x)^{75}$

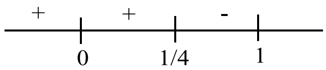
$\Rightarrow \dfrac{dy}{dx} = 25x^{24}(1-x)^{75} - 75x^{25}(1-x)^{74}$

$= 25x^{24}(1-x)^{74}(1 - x - 3x)$

$= 25x^{24}(1-x)^{74}(1-4x)$

Clearly, critical points are 0, 1/4 and 1

Sign scheme of $\frac{dy}{dx}$



Thus, $x = 1/4$ is the point of maxima

107 **(b)**

We have $f(x) = a\log|x| + bx^2 + x$

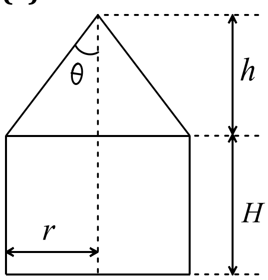
$\Rightarrow f'(x) = \frac{a}{x} + 2bx + 1$

Since, $f(x)$ attains its extremum values at $x = -1, 2$

$\Rightarrow f'(-1) = 0$ and $f'(2) = 0$

$\Rightarrow -a - 2b + 1 = 0$ and $\frac{a}{2} + 4b + 1 = 0 \Rightarrow a = 2$ and $b = -1/2$

108 **(a)**



Given volume and r

Now, $V =$ volume of cone + volume of cylinder

$= \frac{\pi}{3}r^2h + \pi r^2 H$

$V = \frac{\pi}{3}r^2(h + 3H) \Rightarrow H = \frac{\frac{3V}{\pi r^2} - h}{3}$

Now, surface area, $S = \pi rl + 2\pi rH =$

$\pi r\sqrt{h^2 + r^2} + 2\pi r \times \left(\frac{\frac{3V}{\pi r^2} - h}{3}\right)$

Now, let $\frac{dS}{db} = 0 \Rightarrow \pi r \frac{h}{\sqrt{h^2 + r^2}} - \frac{2\pi r}{3} = 0$

$\Rightarrow \frac{h}{\sqrt{h^2 + r^2}} = \frac{2}{3} \Rightarrow 5h^2 = 4r^2 \Rightarrow \frac{r}{h} = \frac{\sqrt{5}}{2} = \tan\theta$

$\Rightarrow \cos\theta = \frac{2}{3} \Rightarrow \theta = \cos^{-1}\frac{2}{3}$

109 **(c)**

The dimensions of the box after cutting equal squares of side x on the corner will be $21 - 2x, 16 - 2x$ and height x

$V = x(21 - 2x)(16 - 2x)$

$= x(336 - 74x + 4x^2)$

or $V = 4x^3 + 336x - 74x^2 \Rightarrow \frac{dV}{dx} = 12x^2 + 336 - 148x$

$\Rightarrow \frac{dV}{dx} = 0$ gives $x = 3$ for which $\frac{d^2V}{dx^2}$ is –ve and hence maximum

110 **(a)**

$f(x) = \sin\left(x + \frac{\pi}{6}\right) + \cos\left(x + \frac{\pi}{6}\right)$

$= \sqrt{2}\sin\left(x + \frac{\pi}{6} + \frac{\pi}{4}\right)$

$= \sqrt{2}\sin\left(x + \frac{5\pi}{12}\right)$

Its maximum value $= \sqrt{2}$ when $x = \frac{5\pi}{12} = \frac{\pi}{2}$

i.e., when $x = \frac{\pi}{2} - \frac{5\pi}{12} = \frac{6\pi - 5\pi}{12} = \frac{\pi}{12}$

111 **(c)**

$\phi'(x) = 2f(x)f'(x)$

We do not know the sign of $f(x)$ in (a, b), so we cannot say about the sign of $\phi'(x)$

112 **(d)**

$y = \frac{\log x}{x}$

$\Rightarrow \frac{dy}{dx} = -\frac{1}{x^2}\log x + \frac{1}{x}\frac{1}{x}$

$= \frac{1}{x^2}(1 - \log x) = 0$

$\frac{dy}{dx} = 0 \Rightarrow \log x = 1$ or $x = e$

For $x < e \Rightarrow \log x < 1$

and $x > e \Rightarrow \log x > 1$

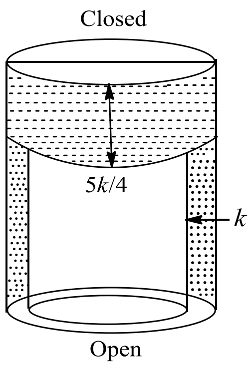
At $x = e, \frac{dy}{dx}$ changes sign from +ve to –ve and hence y is maximum at $x = e$ and its value is $\frac{\log e}{e} = e^{-1}$

113 **(d)**

The derivative of a degree 3 polynomial is quadratic. This must have either 0, 1 or 2 roots. If this has precisely one root, then this must be repeated. Hence, we have $f'(x) = m(x - \alpha)^2$, where α is the repeated root and $m \in R$. So, our original function f has a critical point at $x = \alpha$

Also, $f''(x) = 2m(x - \alpha)$, in which case $f''(\alpha) = 0$. But we are told that the 2nd derivative is non-zero at critical point. Hence, there must be either 0 or 2 critical points

114 **(c)**



Let x be the radius and y the height of the cylinder gas container. Also let k be the thickness of the plates forming the cylindrical sides. Therefore, the thickness of the plate forming the top will be $5k/4$

Capacity of the vessel = vol. of cylinder

$= \pi x^2 y = V$ (Given) $\Rightarrow y = V/(\pi x^2)$ (1)

Now, the volume V_1 of the iron plate used for construction of the container is given by

$V_1 = \pi(x+k)^2(y+5k/4) - \pi x^2 y$

$\Rightarrow \frac{dV_1}{dx} = 2Vk(x+k) \times \left(\frac{5\pi}{4V} - \frac{1}{x^3}\right)$

For maximum or minimum of V_1, $dV_1/dx = 0$

$\Rightarrow x = [4V/(5\pi)]^{1/3}$

For this value of x, d^2V_1/dx^2 is +ve

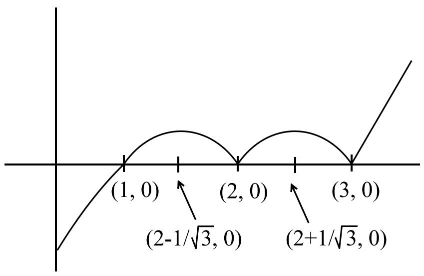
Hence, V_1 is minimum when $x = [4V/(5\pi)]^{1/3}$

Now $x = [4V/(5\pi)]^{1/3}$

$\Rightarrow 5\pi x^3 = 4V = 4\pi x^2 y \Rightarrow x/y = 4/5$

Hence, the required ratio is 4:5

115 **(a)**



$f(x) = (x-1)|(x-2)(x-3)|$

Let $g(x) = (x-1)(x-2)(x-3)$

$= x^3 - 6x^2 + 11x - 6$

$\Rightarrow g'(x) = 3x^2 - 12x + 11$

$g'(x) = 0 \Rightarrow \frac{12 \pm \sqrt{144-132}}{6} = \frac{12 \pm \sqrt{12}}{6}$

$= 2 \pm \frac{1}{\sqrt{3}}$

Hence, $f(x)$ decreases in $\left(2 - \frac{1}{\sqrt{3}}, 2\right) \cup \left(2 + \frac{1}{\sqrt{3}}, 3\right)$

116 **(c)**

Then given expression is minimum when $y = (x^2-3)^3 + 27$ is minimum, which is so when $x = 0$

Hence $y_{\min} = 0$

$\Rightarrow$ Min. value of $2^{(x^2-3)^3+27}$ is $2^0 = 1$

117 **(a)**

$2x^2 + y^2 = 12 \Rightarrow \frac{dy}{dx} = -\frac{2x}{y}$

Slope of normal at point $A(2,2)$ is $\frac{1}{2}$

Also point $B\left(-\frac{22}{9}, -\frac{2}{9}\right)$ lies on the curve and

slope of AB is $= \frac{2-(-2/9)}{2-(-22/9)} = \frac{1}{2}$

Hence the normal meets the curve again at point $\left(-\frac{22}{9}, -\frac{2}{9}\right)$

118 **(b)**

$f(0) = \sin 0 = 0$

$f(0^+) \to 0^+$

$f(0^-) = \lim_{x\to 0^-} \sin(x^2 - 3x)$

$= \lim_{h\to 0} \sin(h^2 + 3h) \to 0^+$

Thus, $f(0^+) > f(0)$ and $f(0^-) > f(0)$

Hence, $x = 0$ is a point of minima

119 **(a)**

$f'(x) = |x| - \{x\} = |x| - (x - [x])$

$= |x| - x + [x]$

For $x \in (-1/2, 0)$,

$f'(x) = -x - x - 1 = -2x - 1$

Also, for $-\frac{1}{2} < x < 0 \Rightarrow 0 < -2x < 1 \Rightarrow -1 < -2x - 1 < 0$

$\Rightarrow f'(x) < 0 \Rightarrow f(x)$ decreases in $(-1/2, 0)$

Similarly, we can check for other given options say for $x \in (-1/2, 2)$

$$f'(x) = \begin{cases} (-x) - x - 1, & -\frac{1}{2} < x < 0 \\ x - x + 0, & 0 \le x < 1 \\ x - x + 1, & 1 \le x < 2 \\ \vdots & \end{cases}$$

Here $f(x)$ decreases only in $(-1/2, 0)$, otherweise $f(x)$ in other intervals is constant

120 **(b)**

Since $f(x)$ has a relative minimum at $x = 0$, therefore $f'(0) = 0$ and $f''(0) > 0$

If the function $y = f(x) + ax + b$ has a relative minimum at $x = 0$, then

$\frac{dy}{dx} = 0$ at $x = 0 \Rightarrow f'(x) + a = 0$ for $a = 0$

$\Rightarrow f'(0) + a = 0 \Rightarrow 0 + a = 0$ $[\because f'(0) = 0] \Rightarrow a = 0$

Now, $\frac{d^2y}{dx^2} = f''(x) \Rightarrow \left(\frac{d^2y}{dx^2}\right)_{x=0} = f''(0) > 0$ $[\because f''(0) > 0]$

Hence, y has a relative minimum at $x = 0$ if $a = 0$ and b can attain any real value

121 **(a)**

$y = x + \sin x \Rightarrow$ If $\frac{dy}{dx} = 1 + \cos x = 0$, then $\cos x = -1$

$\Rightarrow x = \pm\pi, \pm 3\pi \dots$

Also $y = \pm\pi, \pm 3\pi \dots$

But for the given constraint on x and y, no such y exists. Hence, no such tangent exists

122 **(b)**

We have $f(x) = (x+1)^{1/3} - (x-1)^{1/3}$

$\therefore f'(x) = \frac{1}{3}(x+1)^{\frac{-2}{3}} - \frac{1}{3}(x-1)^{\frac{-2}{3}}$

$= \frac{(x-1)^{2/3} - (x+1)^{2/3}}{3(x^2-1)^{2/3}}$

Clearly $f'(x)$ does not exist at $x = \pm 1$

Now, $f'(x) = 0$

$\Rightarrow (x-1)^{2/3} = (x+1)^{2/3}$

$\Rightarrow (x-1)^2 = (x+1)^2$

$\Rightarrow -2x = 2x \Rightarrow 4x = 0 \Rightarrow x = 0$

Clearly, $f'(x) \neq 0$ for any other values of $x \in [0,1]$

The value of $f(x)$ at $x = 0$ is 2

Hence, the greatest value of $f(x) = 2$

123 **(d)**

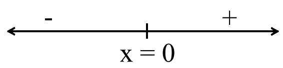
We have $f(x) = \frac{x^2-a}{x^2+a} = 1 - \frac{2a}{x^2+a}$

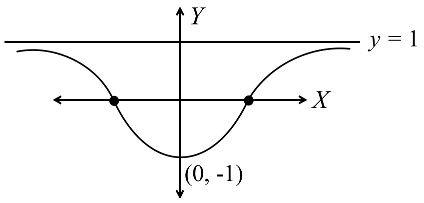
Clearly range of f is $[-1, 1)$

Now, $f'(x) = \frac{4ax}{(x^2+a)^3}$

and $f''(x) = \frac{4a}{(x^2+a)^3}(a - 3x^2)$

Sign scheme of $f'(x)$





$\Rightarrow f(x)$ is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$

Therefore, $f(x)$ has a local minimum at $x = 0$

124 **(b)**

$\because$ Fuel charges $\propto v^2$. Let F represents fuel charges

$\Rightarrow F \propto v^n \Rightarrow F = kv^2 \quad (1)$

Given that $F =$ Rs. 48 per hour, $v = 16$ km per hour

$\Rightarrow 48 = k(16)^2 \Rightarrow k = \frac{3}{16}$

From (1), $F = \frac{3v^2}{16}$

Let the train covers λ km in t hours

$\Rightarrow \lambda = vt$ or $t = \frac{\lambda}{v}$

$\Rightarrow$ Fuel charges in time $t = \frac{3}{16}v^2 \times \frac{\lambda}{v} = \frac{3v\lambda}{16}$

$\Rightarrow$ Total cost for running the train,

$C = \frac{3v\lambda}{16} + 300 \times \frac{\lambda}{v}$

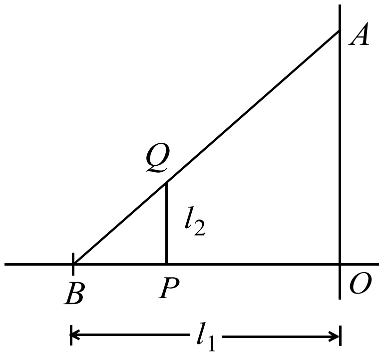
$\Rightarrow \frac{dC}{dv} = \frac{3\lambda}{16} - \frac{300\lambda}{v^2}$ and $\frac{d^2C}{dv^2} = \frac{600\lambda}{v^3}$

For the maximum or minimum value of $C, \frac{dC}{dv} = 0$

$\Rightarrow v = 40$ km/hr. Also, $\left.\frac{d^2C}{dv^2}\right|_{v=40} = \frac{60\lambda}{(40)^3} > 0$ ($\because \lambda > 0$)

$\Rightarrow C$ is minimum when $v = 40$ km/hr

125 **(b)**



Let $BP = x$. From similar triangle property, we get $\frac{AO}{l_1} = \frac{l_2}{x}$

$\Rightarrow AO = \frac{l_1 l_2}{x} \Rightarrow \frac{d(AO)}{dt} = \frac{-l_1 l_2}{x^2}\frac{dx}{dt}$, when

$x = \frac{l_1}{2}, \frac{d(AO)}{dt} = -\frac{2l_2}{5}$ m/s

126 **(a)**

$f(x) + f''(x) = -x\,g(x)f'(x)$

Let $h(x) = f^2(x) + \left(f'(x)\right)^2$

$\Rightarrow h'(x) = 2f(x)f'(x) + 2f'(x)f''(x)$

$= 2f'(x)[-x]g(x)f'(x)$

$= -2x\left(f'(x)\right)^2 g(x)$

$\Rightarrow x = 0$ is a point of maxima for $h(x)$

127 **(b)**

Let $h(x) = f(x) - g(x)$

$h'(x) = f'(x) - g'(x) > 0\ \forall x \in R$

$\Rightarrow h(x)$ is an increasing function and

$h(0) = f(0) - g(0) = 0$

Therefore, $h(x) > 0\ \forall x \in (0, \infty)$ and $h(x) < 0\ \forall x \in (-\infty, 0)$

128 **(b)**

Let $g(x) = 4x^3 - 12x^2 + 11x - 3$

$\Rightarrow g'(x) = 12x^2 - 24x + 11$

$= 12(x-1)^2 - 1$

$\Rightarrow g'(x) > 0$ for $x \in [2, 3]$

$\Rightarrow g(x)$ is increasing in $[2, 3]$

$f(x)_{\max} = f(3) = \log_{10}(4.27 - 12.9 + 11.3 - 3)$

$= \log_{10}(30)$

$= 1 + \log_{10} 3$

129 **(b)**

$g'(x) = xf'(2x^2-1) - xf'(1-x^2)$

$= x\left(f'(2x^2-1) - f'(1-x^2)\right)$

$g'(x) > 0$

If $x > 0, 2x^2 - 1 > 1 - x^2$ (as f' is an increasing function)

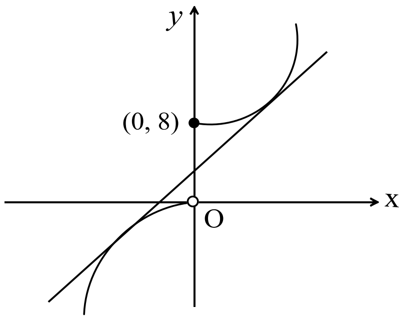
$\Rightarrow 3x^2 > 2 \Rightarrow x \in \left(-\infty, -\sqrt{\frac{2}{3}}\right) \cup \left(\sqrt{\frac{2}{3}}, \infty\right)$

$\Rightarrow x \in \left(\sqrt{\frac{2}{3}}, \infty\right)$

If $x < 0, 2x^2 - 1 < 1 - x^2$

$\Rightarrow 3x^2 < 2 \Rightarrow x \in \left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right) \Rightarrow x \in \left(-\sqrt{\frac{2}{3}}, 0\right)$

130 **(b)**



Let $y = mx + c$ be a tangent to $f(x)$

$y = x^2 + 8$ for $x \geq 0$

$mx + c = x^2 + 8$

$x^2 - mx + 8 - c = 0$ (for the line to be tangent $D = 0$)

$\therefore m^2 = 4(8 - c)$ (1)

Again $y = -x^2$, for $x < 0$

$mx + c = -x^2$

$x^2 + mx + c = 0$

$D = 0 \Rightarrow m^2 = 4c$ (2)

From (1) and (2), we get

$c = 4, m = 4$

$\therefore y = 4x + 4$

Put $y = 0 \Rightarrow x = -1$

131 **(a)**

Here, $f(x) = 4\tan x - \tan^2 x + \tan^3 x$

$\Rightarrow f'(x) = 4\sec^2 x - 2\tan x \sec^2 x + 3\tan^2 x \sec^2 x$

$= \sec^2 x\,(4 - 2\tan x + 3\tan^2 x)$

$= 3\sec^2 x\left\{\tan^2 x - \frac{2}{3}\tan x + \frac{4}{3}\right\}$

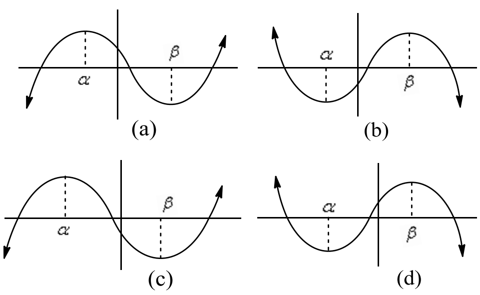
$= 3\sec^2 x\left\{\left(\tan x - \frac{1}{3}\right)^2 + \left(\frac{4}{3} - \frac{1}{9}\right)\right\}$

$= 3\sec^2 x\left\{\left(\tan x - \frac{1}{3}\right)^2 + \frac{11}{9}\right\} > 0, \forall x$

Therefore, $f(x)$ is increasing for all $x \in$ domain

132 **(d)**

From the given data, graph of $f(x)$ can be shown as



Thus from graph, nothing can be said about roots when the sign of $f(\alpha)$ and $f(\beta)$ is given

133 **(d)**

Given, $f(x) = \int_{x^2}^{x^2+1} e^{-t^2} dt$

$\Rightarrow f'(x) = e^{-(x^2+1)^2} 2x - e^{-(x^2)^2}.2x$

$= 2xe^{-(x^4+2x^2+1)}\{1 - e^{2x^2+1}\}$

Here, $e^{2x^2+1} > 1$

And $e^{-(x^4+2x^21)} > 0$for all x

For $f'(x) > 0, x < 0$

134 **(b)**

Given curve is $x^{3/2} + y^{3/2} = 2a^{3/2}$ (1)

$\therefore \frac{3}{2}\sqrt{x} + \frac{3}{2}\sqrt{y}\,\frac{dy}{dx} = 0$ (Differentiate w.r.t. x)

or $\frac{dy}{dx} = -\frac{\sqrt{x}}{\sqrt{y}}$

Since the tangent is equally inclined to the axes

$\therefore \frac{dy}{dx} = \pm 1$

$\therefore -\frac{\sqrt{x}}{\sqrt{y}} = \pm 1 \Rightarrow -\frac{\sqrt{x}}{\sqrt{y}} = -1$ $[\because \sqrt{x} > 0, \sqrt{y} > 0]$

$\Rightarrow \sqrt{x} = \sqrt{y}$

Putting $\sqrt{y} = \sqrt{x}$ in (1), we get

$2x^{3/2} = 2a^{3/2} \Rightarrow x^3 = a^3$

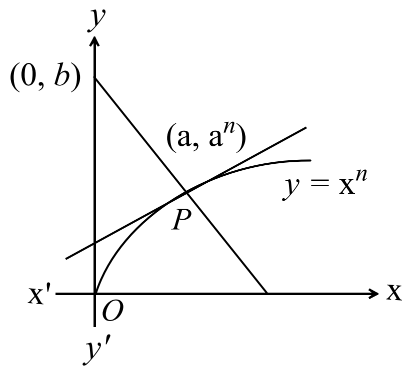
$\therefore x = a$ and so $y = a$

135 **(c)**

$y = x^n$

$\frac{dy}{dx} = n\,x^{n-1} = na^{n-1}$

Slope of the normal $= -\frac{1}{na^{n-1}}$



Equation of the normal $y - a^n = -\frac{1}{na^{n-1}}(x - a)$

Put $x = 0$ to get y-intercept

$y = a^n + \frac{1}{na^{n-2}}$; hence, $b = a^n + \frac{1}{na^{n-2}}$

$\lim_{a\to 0} b = \begin{cases} 0, & \text{if } n < 2 \\ \frac{1}{2}, & \text{if } n = 2 \\ \infty, & \text{if } n > 2 \end{cases}$

136 **(b)**

$4x^2 + 9y^2 = 72$

Differentiating w.r.t. x, we have

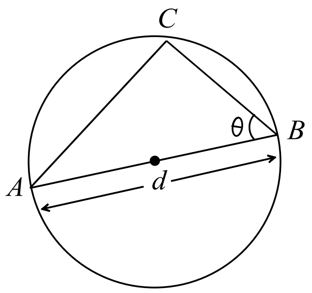
$\Rightarrow 8x + 18y\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{4}{9}\dfrac{x}{y}$

At $(3,2), \frac{dy}{dx} = -\frac{4}{9} \times \frac{3}{2} = -\frac{2}{3}$

Also $x^2 - y^2 = 5 \Rightarrow \frac{dy}{dx} = \frac{x}{y}$. At $(3,2), \frac{dy}{dx} = \frac{3}{2}$

$\therefore$ The curves cut orthogonally

137 **(a)**



Area of $\Delta ABC, A = \frac{1}{2} AC \times BC$

$= \frac{1}{2}(d\sin\theta)(d\cos\theta)$, where $\theta \in (0, \pi/2)$

$= \dfrac{d^2}{4}\sin 2\theta$

Which is maximum when $\sin 2\theta = 1$ or $\theta = \pi/4$

Hence, $AC = BC$, then the triangle is isosceles

138 **(d)**

$y = b\,e^{-x/a}$ meets the y-axis at $(0, b)$

Again $\frac{dy}{dx} = b\,e^{-x/a}\left(-\frac{1}{a}\right)$

At $(0,b), \frac{dy}{dx} = be^0\left(-\frac{1}{a}\right) = -\frac{b}{a}$

$\therefore$ required tangent is $y - b = -\frac{b}{a}(x - 0)$

or $\frac{x}{a} + \frac{x}{b} = 1$

139 **(b)**

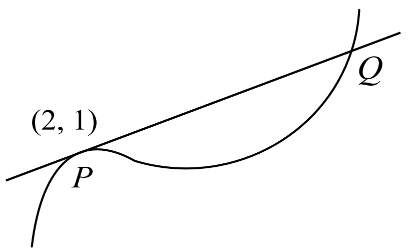
Applying Rolle's theorem to $F(x) = f(x) - 2\,g(x)$, we get $F(0) = 0$,

$F(1) = f(1) - 2g(1)$

$\Rightarrow 0 = 6 - 2g\,(1)$

$\Rightarrow g(1) = 3$

140 **(d)**



Eliminating t gives $y^2(x-1) = 1$

Equation of the tangent at $P(2,1)$ is $x + 2y = 4$

Solving with curve $x = 5$ and $y = -1/2$

$\Rightarrow Q\left(5, -\dfrac{1}{2}\right) \Rightarrow PQ = \dfrac{3\sqrt{5}}{2}$

141 **(c)**

$f(x) = \dfrac{t + 3x - x^2}{x - 4}$; $f'(x)$

$= \dfrac{(x-4)(3-2x) - (t + 3x - x^2)}{(x-4)^2}$

For maximum or minimum, $f'(x) = 0$

$-2x^2 + 11x - 12 - t - 3x + x^2 = 0$

$-x^2 + 8x - (12 + t) = 0$

For one maxima and minima,

$D > 0$

$\Rightarrow 64 - 4(12 + t) = 0$

$\Rightarrow 16 - 12 - t > 0 \Rightarrow 4 > t$ or $t < 4$

142 **(d)**

a. Discontinuous at $x = 1 \Rightarrow$ not applicable

b. $F(x)$ is not continuous (jump discontinuity) at $x = 0$

c. Discontinuity (missing point) at $x = 1 \Rightarrow$ not applicable

d. Notice that $x^3 - 2x^2 - 5x + 6 = (x-1)(x^2 - x - 6)$

Hence, $f(x) = x^2 - x - 6$ if $x \neq 1$ and $f(1) = -6$

$\Rightarrow f$ is continuous at $x = 1$. So $f(x) = x^2 - x - 6$ is continuous in the interval $[-2, 3]$

Also, note that $f(-2) = f(3) = 0$. Hence, Rolle's theorem applies $f'(x) = 2x - 1$

Setting $f'(x) = 0$, we obtain $x = 1/2$ which lies between -2 and 3

143 **(d)**

We have $y^2 = 18x$ (1)

$\therefore 2y\dfrac{dy}{dx} = 18 \Rightarrow \dfrac{dy}{dx} = \dfrac{9}{y}$

Given that $\frac{dy}{dx} = 2 \Rightarrow \frac{9}{y} = 2 \Rightarrow y = \frac{9}{2}$

Putting in (1), we get $\frac{81}{4} = 18x \Rightarrow x = \frac{9}{8}$

Hence, the point is $\left(\frac{9}{8}, \frac{9}{2}\right)$

144 **(a)**

$y = \dfrac{2}{3}x^3 + \dfrac{1}{2}x^2$

$\therefore \dfrac{dy}{dx} = \dfrac{2}{3}3x^2 + \dfrac{1}{2}2x = 2x^2 + x$

Since the tangent makes equal angles with the axes

$\Rightarrow \dfrac{dy}{dx} = \pm 1$

$\Rightarrow 2x^2 + x = \pm 1$

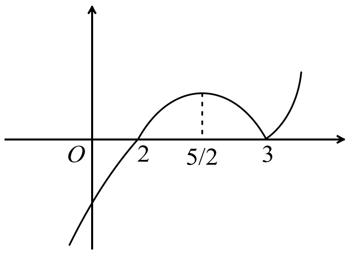
$\Rightarrow 2x^2 + x - 1 = 0$ ($2x^2 + x + 1 = 0$ has no real roots)

$\Rightarrow (2x - 1)(x + 1) = 0$

$\Rightarrow x = \dfrac{1}{2}$ or $x = -1$

145 **(a)**

$f(x) = (x-2)|x-3|$
For, $f(x) = (x-2)(x-3) = x^2 - 5x + 6$
$f'(x) = 2x - 5 = 0 \Rightarrow x = 5/2$
Now, the graph of $f(x) = (x-2)|x-3|$ is



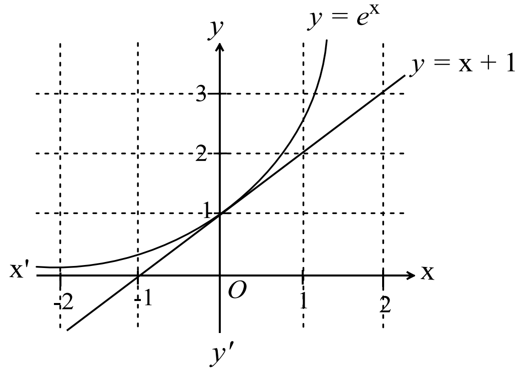
Clearly from the graph, $f(x)$ increases in $(-\infty, 5/2) \cup (3, \infty)$

146 **(d)**
Given $4x + 2\pi r = a$
Where x is side length of the square and r is radius of the circle
$$A = x^2 + \pi r^2 = \frac{1}{16}(a - 2\pi r)^2 + \pi r^2$$
$\frac{dA}{dr} = 0$ gives $r = \frac{a}{2(\pi+4)}$ for which $\frac{d^2A}{dr}$ is +ve and hence minimum
$$\Rightarrow 4x = a - 2\pi r = a - \frac{a\pi}{\pi+4} = \frac{4a}{\pi+4}$$
$$\therefore x = \frac{a}{\pi+4}$$
$$\therefore A = x^2 + r^2\pi = \frac{a^2}{4(\pi+4)}$$

147 **(a)**
$$V = \frac{4}{3}\pi r^3; S = 4\pi r^2$$
$$\frac{dV}{dr} = 4\pi r^2, \frac{dS}{dr} = 8\pi r$$
$$\Rightarrow \frac{dV}{dS} = \frac{dV/dr}{dS/dr} = \frac{4\pi r^2}{8\pi r} = \frac{r}{2}$$
When $r = 2, \frac{dV}{dS} = \frac{2}{2} = 1$

148 **(d)**
When $f(x) = 3x^2 - 2x + 1$
$\therefore f'(x) = 6x - 2$
f is increasing $\Rightarrow f'(x) \geq 0$
$$\Rightarrow 6x - 2 \geq 0 \Rightarrow x \geq \frac{1}{3}$$

149 **(b)**
$$y = e^x \Rightarrow \frac{dy}{dx} = e^x$$
Then, equation of the tangent at $x = 0$ is
$y - 1 = 1(x - 0)$ or $y = x + 1$



Graph of $y = e^x$ always lies above the graph of $y = 1 + x$,
Hence, $e^x > 1 + x \Rightarrow x > \log_e(1+x)$. Hence, **b**. is true
c. is wrong as $\sin x < x$ for $x \in (0,1)$
and **d**. is wrong as $x > \log_e x$ for $\forall x > 0$

150 **(c)**
$$f'(x) = \frac{0.6(1+x)^{-0.4}(1+x^{0.6}) - 0.6x^{-0.4}(1+x)^{0.6}}{(1+x^{0.6})^2}$$
$$= 0.6\frac{(1+x^{0.6}) - x^{-0.4}(1+x)^1}{(1+x^{0.6})^2(1+x)^{0.4}}$$
$$= 0.6\frac{(1+x^{0.6})x^{0.4} - (1-x)}{(1+x^{0.6})^2(1+x)^{0.4}x^{0.4}}$$
$$= 0.6\frac{x^{0.4} - 1}{(1+x^{0.6})^2(1+x)^{0.4}x^{0.4}} < 0 \quad \forall x \in (0,1)$$
Hence, $f(x)$ is decreasing
$\Rightarrow f(x)_{\max} = f(0) = 1$

151 **(a)**
Let $f(x) = x + \frac{1}{x}$
$\therefore f'(x) = 1 - \frac{1}{x^2}$ and $f''(x) = \frac{2}{x^3}$
For maximum/minimum, $f'(x) = 0 \Rightarrow 1 - \frac{1}{x^2} = 0$
$\Rightarrow x^2 = 1 \Rightarrow x \pm 1$
$f(x)$ is minimum at $x = 1$ $\left[\because f''(x) = \frac{2}{1} = 2 > 0\right]$

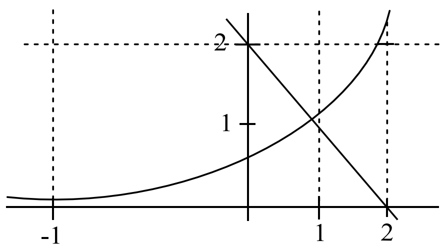
152 **(b)**
Using Lagrange's mean value theorem for f in [1, 2]
for $c \in (1,2), \frac{f(2) - f(1)}{2-1} = f'(c) \leq 2$
$\Rightarrow f(2) - f(1) \leq 2$
$\Rightarrow f(2) \leq 4 \quad (1)$
Again using Lagrange's mean value theorem in [2, 4]
for $d \in (1,2), \frac{f(4) - f(2)}{4-2} = f'(d) \leq 2$
$\Rightarrow f(4) - f(2) \leq 4$
$\Rightarrow 8 - f(2) \leq 4$
$\Rightarrow f(2) \geq 4 \quad (2)$
From (1) and (2), $f(2) = 4$

153 **(b)**

Let $f(x) = x + ax^{-2} - 2$
$\Rightarrow f'(x) = 1 - 2ax^{-3} = 0 \Rightarrow x = (2a)^{1/3}$
Also $f''(x) = 6ax^{-4} \Rightarrow f''\left((2a)^{1/3}\right) > 0$
$\Rightarrow x = (2a)^{1/3}$ is the point of minima
For $x + ax^{-2} - 2 > 0\ \forall x$ we must have
$f\left((2a)^{1/3}\right) > 0$
$\Rightarrow (2a)^{1/3} + a(2a)^{-2/3} - 2 > 0$
$\Rightarrow 2a + a - 2(2a)^{2/3} > 0$
$\Rightarrow 3a > 2(2a)^{2/3}$
$\Rightarrow 27a^3 > 32a^2$
$\Rightarrow a > 32/37$
Hence, the least value of a is 2

154 **(a)**
$f'(x) = ax^2 + 2(a+2)x + (a-1)$
$f''(x) = 2ax + 2(a+2) = 0$
$\Rightarrow x = -\frac{a+2}{a}$ which is the point of inflection
Given that, we must have $-\frac{a+2}{a} < 0$
$\Rightarrow (-\infty, -2) \cup (0, \infty)$

155 **(a)**
Let $f(x) = e^{x-1} + x - 2 \Rightarrow f'(x) = e^{x-1} + 1 > 0\ \forall x \in R$
Also when $x \to \infty, f(x) \to \infty$ and when $x \to \infty, f(x) \to \infty$
Further $f(x)$ is continuous, hence its graph cuts x-axis only at one point
Hence, equation $f(x) = 0$ has only one root
Alternative method:
Also $e^{x-1} = 2 - x$
As shown in the figure, graphs of $y = e^{x-1}$ and $y = 2 - x$ cuts at only one point. Hence, there is only one root



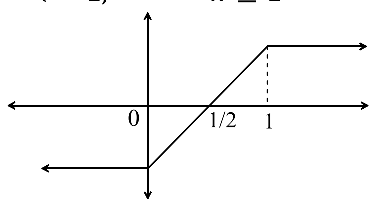
156 **(a)**
$$g(x) = \frac{x+2}{x-1}$$
$$\Rightarrow g'(x) = \frac{-3}{(x-1)^2}$$
Slope of given line $= -3 \Rightarrow = \frac{-3}{(x-1)^2} = -3$
$\Rightarrow x = 2$, also $g(2) = 4$
$(2, 4)$ also lies on given line
Hence the given line is tangent to the curve

157 **(c)**
We have $f(x) = \frac{x}{\sin x}, 0 < x \le 1$
$$\Rightarrow f'(x) = \frac{\sin x - x\cos x}{\sin^2 x}$$
$$= \frac{\cos x\,(\tan x - x)}{\sin^2 x}$$
We know that $\tan x > x$ for $0 < x < \pi/2$
$\Rightarrow f'(x) > 0$ for $0 < x <\le 1$
Hence, $f(x)$ is an increasing function
$$g(x) = \frac{x}{\tan x}$$
$$\Rightarrow g'(x) = \frac{\tan x - x\sec^2 x}{\tan^2 x} = \frac{\sin x\cos x - x}{\sin^2 x}$$
$$= \frac{\sin 2x - 2x}{2\sin^2 x}$$
$= \frac{\sin\theta - \theta}{2\sin^2(\theta/2)}$, where $\theta \in (0, 2)$
We know that $\sin\theta < \theta$ for $\forall\ \theta > 0$
$\Rightarrow g'(x) < 0 \Rightarrow g(x)$ is a decreasing function

158 **(c)**
It is a fundamental property

159 **(b)**
We have $f''(x) = 6(x-1)$
Integrating, we get $f'(x) = 3(x-1)^2 + c$ (1)
At $(2, 1)$, $y = 3x - 5$ is tangent to $y = f(x)$
$\therefore f'(2) = 3$
From equation (1), $3 = 3(2-1)^2 + c \Rightarrow 3 = 3 + c \Rightarrow c = 0$
$\therefore f'(x) = 3(x-1)^2$
Integrating, we get $f(x) = (x-1)^3 + c'$
Since the curve passes through $(2, 1)$
$\therefore 1 = (2-1)^3 + c' \Rightarrow c' = 0$
$\therefore f(x) = (x-1)^3$
$\therefore f(0) = -1$

160 **(d)**
$f(x) = |x| - |x-1|$
$$= \begin{cases} -x - (1-x), & x < 0 \\ x - (1-x), & 0 \le x < 1 \\ x - (x-1), & x \ge 0 \end{cases}$$
$$= \begin{cases} -1, & x < 0 \\ 2x - 1, & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases}$$



Graph of the function is that $f(x)$ clearly increases in $(0, 1)$

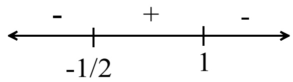
161 **(c)**
$$f(x) = \frac{(\sin x + \cos x)^2 - 1}{\frac{1}{\sqrt{2}}(\sin x + \cos x)} = \sqrt{2}\frac{t^2 - 1}{t}$$
or $f(x) = \phi(t) = \sqrt{2}\left(t - \frac{1}{t}\right)$

Where $t = g(x) = \sin x + \cos x, x \in [0, \pi/2]$
$g'(x) = \cos x - \sin x = 0 \Rightarrow \tan x = 1$
$\Rightarrow x = \pi/4$ and $g''(x) = -\text{ve}$
At $x = 0, t = 1 \therefore t \in [1, \sqrt{2}]$
Now $\phi(t) = \sqrt{2}\left(t - \frac{1}{t}\right)$ where $t \in [1, \sqrt{2}]$
$$\phi'(t) = \sqrt{2}\left(1 + \frac{1}{t^2}\right) = +\text{ ve}$$
Therefore, $\phi(t)$ is increasing
Hence $\phi(t)$ is greatest at the endpoint of interval $[1, \sqrt{2}]$ i.e. $t = \sqrt{2}$
$$\therefore f(x) = \phi(t) = \sqrt{2}\left[\sqrt{2} - \frac{1}{\sqrt{2}}\right] = 1$$
Alternative method:
$$f(x)\frac{\sin 2x}{\sin\left(x + \frac{\pi}{4}\right)} = \frac{2\sin x \cos x}{\frac{1}{\sqrt{2}}(\sin x + \cos x)}$$
$$= 2\sqrt{2}\frac{1}{\sec x + \operatorname{cosec} x}$$
For $x \in (0, \pi/2)$, maximum value of $\sec x + \operatorname{cosec} x$ occurs when $\sec x = \operatorname{cosec} x$ or $x = \pi/4$
Hence, $f_{\max} = \frac{2\sqrt{2}}{\sec\frac{\pi}{4} + \operatorname{cosec}\frac{\pi}{4}} = \frac{2\sqrt{2}}{2\sqrt{2}} = 1$

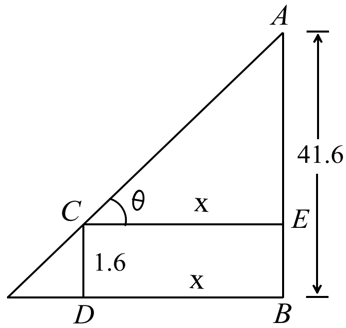
162 **(b)**
$$x^2 - y^2 = 8 \Rightarrow \frac{dy}{dx} = \frac{x}{y} \Rightarrow -\frac{1}{dy/dx} = -\frac{y}{x}$$
At the point $\left(-\frac{5}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right), -\frac{1}{dy/dx} = \frac{-3/\sqrt{2}}{-5/\sqrt{2}} = \frac{3}{5}$
Also $9x^2 + 25y^2 = 225$
$$\Rightarrow 18x + 50y\frac{dy}{dx} = 0$$
$$\Rightarrow \frac{dy}{dx} = -\frac{9x}{25y} \Rightarrow -\frac{dx}{dy} = \frac{25y}{9x}$$
At the point $\left(-\frac{5}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$
$$-\frac{dx}{dy} = \frac{25 \times 3/\sqrt{2}}{9\,(-5/\sqrt{2})} = -\frac{15}{9} = -\frac{5}{3}$$
Since the product of the slopes $= -1$. Therefore, the normals cut orthogonally, i.e., the required angle is equal to $\frac{\pi}{2}$

163 **(a)**
$$h'(x) = \frac{m}{n}x^{\frac{m-n}{n}} = \frac{m}{n}x^{-\left(\frac{\text{even}}{\text{odd}}\right)}$$
As $h'(x)$ is undefined at $x = 0$ and $h'(x)$ does not change its sign in the neighbourhood. So, no extremums

164 **(d)**
Let $g(x) = f(x) - x^2$. We have $g(1) = 0, g(2) = 0, g(3) = 0$
$[\because f(1) = 1, f(2) = 4, f(3) = 9]$
From Rolle's theorem on $g(x)$, $g'(x) = 0$ for at least $x \in (1, 2)$. Let $g'(c_1) = 0$ where $c_1 \in (1, 2)$
Similarly, $g(x) = 0$ for at least one $x \in (2, 3)$. Let $g'(c_2) = 0$ where $c_2 \in (2, 3)$
$\therefore g'(c_1) = g'(c_2) = 0$
By Rolle's theorem, at least one $x \in (c_1, c_2)$ such that $g''(x) = 0 \Rightarrow f''(x) = 2$ for some $x \in (1, 3)$

165 **(a)**
$f(x) = xe^{x(1-x)}$
$\Rightarrow f'(x) = e^{x(1-x)} + (1 - 2x)xe^{x(1-x)}$
$= -e^{x(1-x)}(2x^2 - x - 1)$
$= -e^{x(1-x)}(2x + 1)(x - 1)$
Sign scheme of $f'(x)$


$\therefore g(x)$ is increasing in $[-1/2, 1]$

166 **(a)**
Let CD be the position of man at any time t. Let $BD = x$, then $EC = x$. Let $\angle ACE = \theta$


Given, $AB = 41.6$ m, $CD = 1.6$ m and $\frac{dx}{dt} = 2$ m/s
$AE = AB - EB = AB - CD = 41.6 - 1.6 = 40$ m
We have to find $\frac{d\theta}{dt}$ when $x = 30$ m
From $\Delta\, AEC, \tan\theta = \frac{AE}{EC} = \frac{40}{x}$ (1)
Differentiating w.r.t. to t, $\sec^2\theta\,\frac{d\theta}{dt} = \frac{-40}{x^2}\frac{dx}{dt}$
$$\Rightarrow \sec^2\theta\,\frac{d\theta}{dt} = \frac{-40}{x^2} \times 2$$
$$\Rightarrow \frac{d\theta}{dt} = \frac{-80}{x^2}\cos^2\theta$$
$$= -\frac{80}{x^2}\frac{x^2}{x^2 + 40^2}\left[\because \cos\theta = \frac{x}{\sqrt{x^2 + 40^2}}\right]$$
$\Rightarrow \frac{d\theta}{dt} = -\frac{80}{x^2 + 40^2}$ (2)
When $x = 30$ m, $\frac{d\theta}{dt} = -\frac{80}{30^2 + 40^2} = -\frac{4}{125}$ radian/s

167 **(a)**
$$f(x) = x\sqrt{4ax - x^2}$$
$$\Rightarrow f'(x) = \sqrt{4ax - x^2} + \frac{x(4a - 2x)}{2\sqrt{4ax - x^2}}$$
$$= \frac{2x(3a - x)}{\sqrt{4ax - x^2}}$$
Now if $f'(x) > 0$
$\Rightarrow 2x(3a - x) > 0$

$\Rightarrow 2x(x-3a)<0$
$\Rightarrow x \in (0, 3a)$
Thus, $f(x)$ increases in $(0, 3a)$ and decreases in $(-\infty, 0) \cup (3a, 4a)$

168 **(a)**

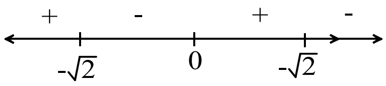
Slope of the tangent at $(x, f(x))$ is $2x+1$
$\Rightarrow f'(x) = 2x+1$
$\Rightarrow f(x) = x^2 + x + c$
Also the curve passes through (1, 2). Therefore, $f(1) = 2$
$\Rightarrow 2 = 1 + 1 + c \Rightarrow c = 0 \Rightarrow f(x) = x^2 + x$
$\Rightarrow$ Required area $= \int_0^1 (x^2 + x)dx$
$= \left(\frac{x^3}{3} + \frac{x^2}{2}\right)_0^1 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$

169 **(b)**

$y = e^x + e^{-x} \Rightarrow \frac{dy}{dx} = e^x - e^{-x} = \tan\theta$, where θ is the angle of the tangent with the x-axis
For $\theta = 60°$, we have $\tan 60° = e^x - e^{-x}$
$\Rightarrow e^{2x} - \sqrt{3}\, e^x - 1 = 0$
$\Rightarrow e^x = \frac{\sqrt{3} \pm \sqrt{7}}{2} \Rightarrow x = \log_e\left(\frac{\sqrt{3}+\sqrt{7}}{2}\right)$

170 **(d)**

$f(x) = x^4 e^{-x^2} \Rightarrow f''(x)$
$= 4x^3 e^{-x^2} + x^4 e^{-x^2}(-2x)$
$= 2x^3 e^{-x^2}(z - x^2)$
Sign scheme of $f''(x)$



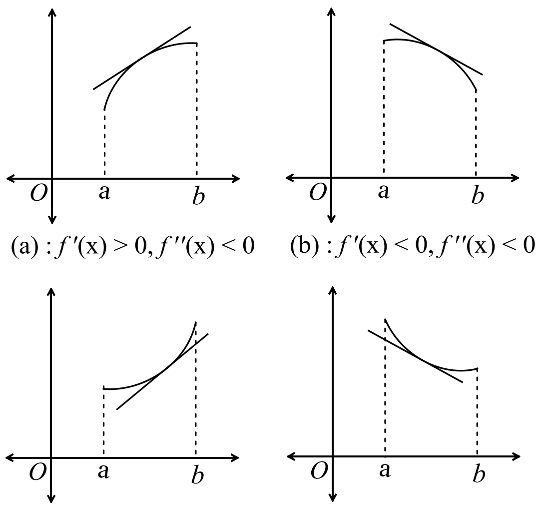
Hence, $f(x)$ is maximum at $x = \pm\sqrt{2} \Rightarrow$ Maximum value $= 4e^{-2}$

171 **(d)**

Solving the curves, we get point of intersection (a^2, a)
For $x = y^2, \frac{dy}{dx} = \frac{1}{2y}$
At $(a^2, a), \frac{dy}{dx} = \frac{1}{2a}$
For $xy = a^3, \frac{dy}{dx} = -\frac{y}{x}$
At $(a^2, a), \frac{dy}{dx} = -\frac{a}{a^2} = -\frac{1}{a}$
Since the curves cut orthogonally
$\therefore \frac{1}{2a} \times -\frac{1}{a} = -1 \Rightarrow 2a^2 = 1 \Rightarrow a^2 = \frac{1}{2}$

172 **(c)**

$f'(x) < 0, f''(x) < 0$



(a) : $f'(x) > 0, f''(x) < 0$ (b) : $f'(x) < 0, f''(x) < 0$
(c) : $f'(x) > 0, f''(x) > 0$ (d) : $f'(x) < 0, f''(x) > 0$

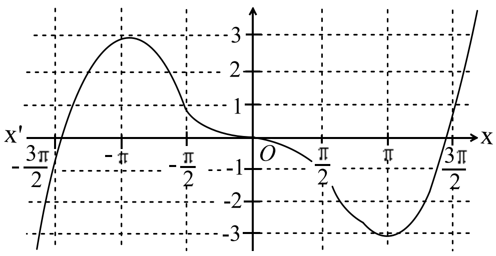
Clearly for $f'(x) > 0, f''(x) > 0$ [in figure] tangent always lies below the graph
Or $f'(x) < 0, f''(x) > 0$ [in figure (d)) tangent always lies below the graph

173 **(a)**

$f(x) = (4\sin^2 x - 1)^n (x^2 - x + 1)$
$x^2 - x + 1 > 0\ \forall x$
$f\left(\frac{\pi}{6}\right) = 0$
$f\left(\frac{\pi^+}{6}\right) = \lim_{x \to \frac{\pi^+}{6}} (4\sin^2 x - 1)^n (x^2 - x + 1)$
$= \to 0^+$
$f\left(\frac{\pi^-}{6}\right) = \lim_{x \to \frac{\pi^-}{6}} (4\sin^2 x - 1)^n (x^2 - x + 1)$
$= (\to 0^-)^n$ (a positive value)
$f\left(\frac{\pi^-}{6}\right) > 0$ if n is an even number

174 **(b)**

$f'(x) = -x\sin x = 0$ when $x = 0$ or π
$f'(0^-) = (-)(-)(-) < 0$
$f'(0^+) = (-)(+)(+) < 0$ } no sign change



This also implies that f is decreasing at $x = 0$
$\Rightarrow (b)$ is correct
$f''(x) = -(x\cos x + \sin x)$
$f''(\pi) = -(-\pi) > 0$ minima at $x = \pi$
$f''(-\pi) = -(\pi) < 0$ maxima at $x = -\pi$

175 **(b,d)**

$f'(x) = \frac{\sin x}{x}$
For $f'(x) = 0, \frac{\sin x}{x} = 0 \Rightarrow x = n\pi (n \in I, n \neq 0)$

$$f''(x) = \frac{x\cos x - \sin x}{x^2}$$

$f''(n\pi) = \frac{\cos n\pi}{n\pi} < 0$ if $n = 2k - 1$ and > 0 if $n = 2k, k \in I^+$

Hence, $f(x)$ has local maxima at $x = n\pi$, where $n = 2k - 1$ and local minima at $x = n\pi, n = 2k$, where $k \in I^+$

176 **(a,b)**

Let $P(x, y)$ be a point on the curve In $(x^2 + y^2) = c\tan^{-1}\frac{y}{x}$

Differentiating both sides with respect to x, we get

$\frac{2x+2yy'}{(x^2+y^2)} = \frac{c\,(xy' - y)}{(x^2+y^2)} \Rightarrow y' = \frac{2x+cy}{cx-2y} = m_1$(say)

Slope of $OP = \frac{y}{x} = m_2$ (say) (where O is origin)

Let the angle between the tangents at P and OP be θ

$$\Rightarrow \tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{\frac{2x+cy}{cx-2y} - \frac{y}{x}}{1 + \frac{2xy+cy^2}{cx^2 - 2xy}}\right| = \frac{2}{c}$$

$\Rightarrow \theta = \tan^{-1}\left(\frac{2}{c}\right)$ which is independent of x and y

177 **(b,c)**

$\because$ g(x) is increasing and $f(x)$ is decreasing

$\Rightarrow$ g$(x+1) >$ g$(x-1)$ and $f(x+1) < f(x-1)$

$\Rightarrow f\{$g$(x+1)\} < f\{$g$(x-1)\}$ and g$\{f(x+1)\} <$ g$\{f(x-1)\}$

178 **(a,b,c,d)**

We are given that

$$f(x) = \begin{cases} 3x^2 + 12x - 1, -1 \le x \le 2 \\ 37 - x, \quad 2 < x \le 3 \end{cases}$$

Then in $[-1, 2], f'(x) = 6x + 12$

$f'(x) = 0 \Rightarrow x = -2$

$\Rightarrow f(x)$ decreases in $(-\infty, -2)$ and increases in $(-2, \infty)$

Also $f(2^-) = 3(2)^2 + 12(2) - 1 = 35$

And $f(2^+) = 37 - 2 = 35$

Hence $f(x)$ is continuous

$$f'(x) = \begin{cases} 6x + 12, -1 < x < 2 \\ -1, \quad 2 < x < 3 \end{cases}$$

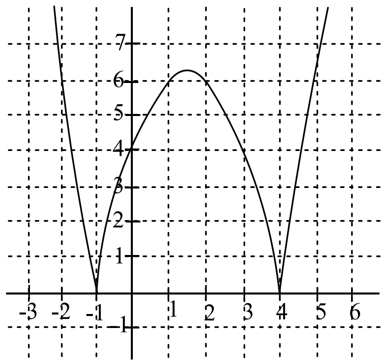
$\Rightarrow f'(2^-) = 24$ and $f'(2^+) = -1$

Hence, $f(x)$ is non-differentiable at $x = 2$

Also, $f(2^+) < f(2)$ and $f(2^-) < f(2)$

Hence, $x = 2$ is the point of maxima

179 **(a,b,c,d)**



Refer the graph for the answers

180 **(a,b,d)**

At the point of inflection, concavity of the curve changes irrespective of any other factor

181 **(b,c)**

$$y = x^2 + 4x - 17 \Rightarrow \frac{dy}{dx} = 2(x+2) \Rightarrow \left(\frac{dy}{dx}\right)_{x=\frac{5}{2}} = 9$$

$\Rightarrow \tan\theta = 9$, where θ is the angle with positive direction of x-axis

$\Rightarrow$ Angle with y-axis is $\frac{\pi}{2} \pm \theta = \frac{\pi}{2} \pm \tan^{-1} 9$

182 **(b,c)**

Here, $f'(x) = \frac{1}{x} + \sqrt{1 + \sin x}, x > 0$ but $f(x)$ is not differentiable in $(0, \infty)$ as $\sin x$ may be-1 and then $f''(x) = -\frac{1}{x^2} + \frac{\cos x}{2\sqrt{1+\sin x}}$ will not exists.

$\Rightarrow \quad f'(x)$ Is continuous for all $x \in (0, \infty)$ but $f'(x)$ is not differentiable on $(0,\infty)$.

$\therefore$ Option (b) is true

Also, $f'(x) \le 3$, if $x > 1$

And $f(x) > 3$, if $x > e^3$

$\therefore$ Let $\alpha = e^3$

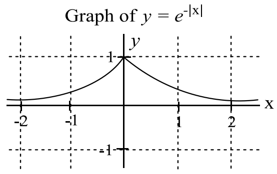
$\Rightarrow$ Option (c) is true.

(d) It is not possible as $f(x) \to \infty$ when$x \to \infty$.

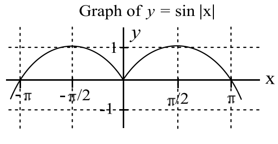
Hence, (b, c) are the correct options.

183 **(a,b,d)**

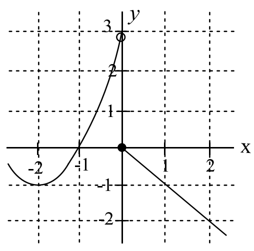
Graph of $y = e^{-|x|}$



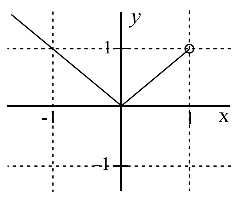
Graph of $y = \sin|x|$



Graph of $f(x) = \begin{cases} x^2 + 4x + 3, x < 0 \\ -x, \quad x \ge 0 \end{cases}$



Graph of $f(x) = \begin{cases} |x|, x < 0 \\ \{x\}, x \ge 0 \end{cases}$



184 **(a)**

Since g is decreasing in $[0, \infty)$

$\therefore$ For $x \geq y \geq 0, g(x) \leq g(y)$ (1)

Also $g(x), g(y) \in [0, \infty)$ and f is increasing from $[0, \infty)$ to $[0, \infty)$

$\therefore$ For $g(x), g(y) \in [0, \infty)$

Such that $g(x) \leq g(y)$

$\Rightarrow f(g(x)) \leq f(g(y))$ where $x \geq y$

$\Rightarrow h(x) \leq h(y)$

$\Rightarrow h$ is a decreasing function from $[0, \infty)$ to $[0, \infty)$

$\therefore h(x) \leq h(0), \forall x \geq 0$

But $h(0) = 0$ (given)

$\therefore h(x) \leq 0, \forall x \geq 0$ (2)

Also $h(x) \geq 0, \forall x \geq 0$ (3)

[as $h(x) \in [0, \infty)$]

From (2) and (3) we get $h(x) = 0, \forall x \geq 0$

Hence, $h(x) - h(1) = 0 - 0 = 0, \forall x \geq 0$

185 **(a,d)**

$f(x) = (\sin^2 x - 1)^n$

$f\left(\frac{\pi}{2}\right) = 0$

$f\left(\frac{\pi^+}{2}\right) = (\to 0^-)^n$ and $f\left(\frac{\pi^-}{2}\right) = (\to 0^-)^n$

If n is even $f\left(\frac{\pi^+}{2}\right)$ and $f\left(\frac{\pi^-}{2}\right) > 0$, then $x = \frac{\pi}{2}$ is the point of minima

If n is odd $f\left(\frac{\pi^+}{2}\right)$ and $f\left(\frac{\pi^-}{2}\right) < 0$, then $x = \frac{\pi}{2}$ is the point of maxima

186 **(a,b,c,d)**

We have $y = ce^{x/a}$

$\Rightarrow \frac{dy}{dx} = \frac{c}{a} e^{x/a} \Rightarrow \frac{dy}{dx} = \frac{1}{a} y$

$\Rightarrow \frac{y}{dy/dx} = a = \text{const.}$

$\Rightarrow$ sub-tangent = const.

$\Rightarrow$ Length of the sub-normal $= y\frac{dy}{dx} = y\frac{y}{a} = \frac{y^2}{a} \propto$ (square of the ordinate)

Equation of the tangent at (x_1, y_1) is $y - y_1 = \frac{y_1}{a}(x - x_1)$

This meets the x-axis at a point given by

$-y_1 = \frac{y_1}{a}(x - x_1) \Rightarrow x = x_1 - a$

The curve meets the y-axis at $(0, c)$

$\therefore \left(\frac{dy}{dx}\right)_{(0,c)} = c/a$

So, the equation of the normal at $(0, c)$ is

$y - c = -\frac{1}{c/a}(x - 0) \Rightarrow ax + cy = c^2$

187 **(c)**

The given polynomial is $p(x) = a_0 + a_1 x^2 + a_2 x^4 + \cdots + a_n x^{2n}, x \in R$ and $0 < a_0 < a_1 < a_2 < \cdots < a_n$

Here we observe that all coefficients of different powers of x, i.e., $a_0, a_1, a_2, \dots a_n$ are positive

Also, only even powers of x are involved

Therefore, $P(x)$ cannot have any maximum value

Moreover, $P(x)$ is minimum, when $x = 0$, i.e., a_0

Therefore, $P(x)$ has only one minimum

Alternative method

We have

$P'(x) = 2a_1 x + 4a_2 x^3 + \cdots + 2na_n x^{2n-1}$

$= x(2a_1 + 4a_2 x^2 + \cdots + 2na_n x^{2n-2})$

Clearly $P'(x) > 0$ for $x > 0$ and $P'(x) < 0$ for $x < 0$

$\Rightarrow P(x)$ increases for all $x > 0$ and decreases for all $x < 0$

Therefore, $P'(x)$ has $x = 0$ as the point of maxima

188 **(b,c)**

Let the line $ax + by + c = 0$ be normal to the curve $xy = 1$

Differentiate the curve $xy = 1$ w.r.t., x, we get

$y + x\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$

$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = -\frac{y_1}{x_1}$

$\therefore$ Slope of the normal $= \frac{x_1}{y_1}$

Slope of the given line $= \frac{-a}{b}$

Given that $\frac{x_1}{y_1} = \frac{-a}{b}$ (1)

Also (x_1, y_1) lies on the given curve $\Rightarrow x_1 y_1 = 1$ (2)

From (1) and (2), we can conclude that a and b must have opposite sign

189 **(a,c)**

We have $h'(x) = f'(x)[1 - 2f(x) + 3f(x)^2]$

$= 3f'(x)\left[(f(x))^2 - \frac{2}{3}f(x) + \frac{1}{3}\right]$

$= 3f'(x)[(f(x) - 1/3)^2 + 2/9]$

Note that $h'(x) < 0$ whenever $f'(x) < 0$ and $h'(x) > 0$ whenever $f'(x) > 0$,

Thus $h(x)$ increases (decreases) whenever $f(x)$ increases (decreases)

$\therefore$ (**a**) and (**c**) are the correct options

190 **(a,b,c)**

$f(x) = \frac{e^x}{1 + e^x}$

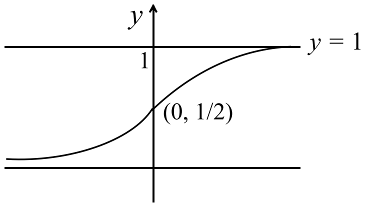
$\Rightarrow f'(x) = \frac{e^x(1 + e^x) - e^x e^x}{(1 + e^x)^2} = \frac{e^x}{(1 + e^x)^2} > 0 \ \forall x \in R$

$\Rightarrow f(x)$ is an increasing function

Also, $\lim_{x \to \infty} \frac{e^x}{1+e^x} = 0$ and

$\lim_{x\to\infty}\frac{e^x}{1+e^x}=\lim_{x\to\infty}\frac{1}{1+\frac{1}{e^x}}=1$

Hence, the graph of $f(x)=\frac{e^x}{1+e^x}$ is as shown



Also, $f'(x)=\frac{e^x(1+e^x)^2-2(1+e^x)e^xe^x}{(1+e^x)^4}=0$

$\Rightarrow (1+e^x)-2e^x=0$

$\Rightarrow e^x=1$

$\Rightarrow x=0$ which is point of inflection

$x=0$ is the inflection point and f is bounded in $(0,1)$

No maximum and f has two asymptotes

191 **(a,d)**

We have

$f(x)=2x+\cot^{-1}x+\log(\sqrt{1+x^2}-x)$

$$\therefore f'(x)=2-\frac{1}{1+x^2}+\frac{1}{\sqrt{1+x^2}-x}\left(\frac{x}{\sqrt{1+x^2}}-1\right)$$

$$=\frac{1+2x^2}{1+x^2}-\frac{1}{\sqrt{1+x^2}}=\frac{1+2x^2}{1+x^2}-\frac{\sqrt{(1+x^2)}}{1+x^2}$$

$=\frac{x^2+\sqrt{1+x^2}(\sqrt{1+x^2}-1)}{1+x^2}>0$ for all x

Hence, $f(x)$ is an increasing function in $(-\infty,\infty)$ and in particular in $(0,\infty)$

192 **(a,b,c)**

$y=x^2\Rightarrow\frac{dy}{dx}=2x=2$ at $(1,1)$

$x=y^2\Rightarrow y=\sqrt{x}$

$\Rightarrow\frac{dy}{dx}=\frac{1}{2\sqrt{x}}=\frac{1}{2}$ at $(1,1)$

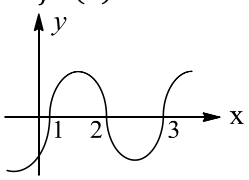
$$\Rightarrow\tan\theta=\frac{2-\frac{1}{2}}{1+2\left(-\frac{1}{2}\right)}=\frac{\frac{3}{2}}{1+1}=\frac{3}{4}$$

$$\Rightarrow\theta=\tan^{-1}\frac{3}{4}=\cos^{-1}\frac{4}{5}=\sin^{-1}\frac{3}{5}$$

193 **(a,b,c)**

Obviously, at $x=0, f(x)=\infty$

$\therefore\ f''(0)$ does not exist



So, $x=0$ is a critical point

$$\text{Now, } f(x)=\begin{cases}\frac{x-2}{x^2}, & x\ge 2\\ \frac{2-x}{x^2}, & 0<x<2\end{cases}$$

At $x=2,4$ the function $f(x)$ is not differentiable, so they are critical points.

194 **(a,b,c)**

$f(x)=a_5x^5+a_4x^4+a_3x^3+a_2x^2+a_1x$

$\Rightarrow f(x)=0$ has one root $x=0$

Also, given that $f(x)=0$ has positive root a_0

Thus, the equation must have at least three real roots (as complex root occurs in conjugate pair).

Thus $f'(x)=0$ has at least two real roots as between two roots of $f(x)=0$, there lies at least one root of $f'(x)=0$

Similarly, we can say that $f''(x)=0$ has at least one real root. Further, $f'(x)=0$ has one root between roots $x=0$ and $x=a_0$ of $f(x)=0$

195 **(a,b)**

Given that $\frac{x^2+x+2}{x^2+5x+6}<0\Rightarrow x\in(-3,-2)$

We have to find the extrema for the function

$f(x)=1+a^2x-x^3$

For maximum or minimum, $f'(x)=0$

$\Rightarrow a^2-3x^2=0$ or $x=\pm\frac{a}{\sqrt{3}}$ and $f''(x)=-6x$ is +ve when x is negative

If a is positive, then the point of minima is $-\frac{a}{\sqrt{3}}$

i.e., $-3<-\frac{a}{\sqrt{3}}<-2$ or $2\sqrt{3}<a<3\sqrt{3}$

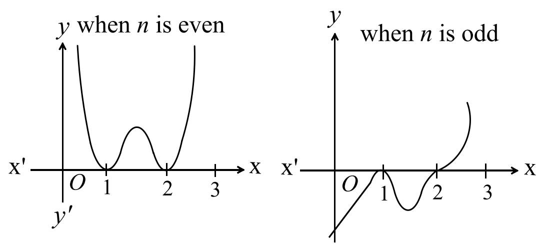
and if a is negative, then the point of minima is $\frac{a}{\sqrt{3}}$

i.e., $-3<\frac{a}{\sqrt{3}}<-2$ or $-3\sqrt{3}<a<-2\sqrt{3}$

Then, $a\in\left(-3\sqrt{3},-2\sqrt{3}\right)\cup(2\sqrt{3},3\sqrt{3})$

196 **(a,c,d)**

Graph of $f(x)$



197 **(a,c,d)**

$$f'(x)=2x-\frac{\lambda}{x^2}\ \therefore f'(x)=0\ \Rightarrow x=\left(\frac{\lambda}{2}\right)^{1/3}$$

If $\lambda=16, x=2$

Now, $f''(x)=2+\frac{2\lambda}{x^3}$

$\therefore$ if $\lambda=16, f''(x)>0$, i.e., $f(x)$ has a minimum at $x=2$

Also, $f''\left\{\left(\frac{\lambda}{2}\right)^{1/3}\right\}=2+\frac{2\lambda}{\lambda/2}=2+4>0$

Hence, $f(x)$ has maximum for no real value of λ

When $\lambda=-1, f''(x)=0$ if $x=1$. So, $f(x)$ has a point of inflection at $x=1$

198 **(a,b,c)**

Let $y = f(x)^{g(x)}$

$\Rightarrow \frac{dy}{dx} = f(x)^{g(x)}\left[g(x)\frac{f'(x)}{f(x)} + g'(x)\log f(x)\right]$

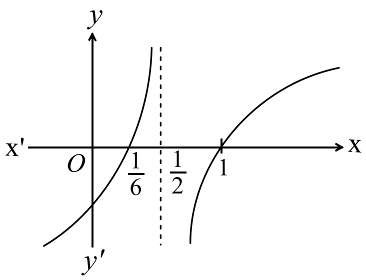
$f(x)^{g(x)}, g(x), f(x), f'(x)$ and $g'(x)$ are positive, but $\log f(x)$ can be negative, which can cause $\frac{dy}{dx} < 0$, hence statement (a) is false

If $f(x) < 1 \Rightarrow \log f(x) < 0$, which does not necessarily make $\frac{dy}{dx} < 0$, hence statement (b) is false

$f(x) < 0$ can also cause $\frac{dy}{dx} > 0$, hence statement (c) is false. But reverse of (c) is true

199 **(a,b,d)**

$f'(x) = \frac{12x^2 - 12x + 5}{(2x-1)^2} > 0 \ \forall x \in R$



Hence f is increasing $\forall x \in R$

$x = 1/2$ is the point of inflection as concavity changes at $x = 1/2$

200 **(b,c)**

$f(x) = x^3 - x^2 + 100x + 2002$

$f'(x) = 3x^2 - 2x + 100 > 0 \ \forall x \in R$

$\therefore f(x)$ is increasing (strictly)

$\therefore f\left(\frac{1}{2000}\right) > f\left(\frac{1}{2001}\right)$

Also, $f(x-1) > f(x-2)$ as $x - 1 > x - 2$ for $\forall x$

201 **(a,d)**

We have $f(x) = (4a-3)(x + \log 5) + 2(a-7)\cot\frac{x}{7}\sin^2\frac{x}{2}$

$= (4a-3)(x + \log 5) + (a-7)\sin x$

$\Rightarrow f'(x) = (4a-3) + (a-7)\cos x$

If $f(x)$ does not have critical points, then $f'(x) = 0$ does not have any solution in R

Now, $f'(x) = 0 \Rightarrow \cos x = \frac{4a-3}{7-a}$

$\Rightarrow \left|\frac{4a-3}{7-a}\right| \le 1 \quad [\because |\cos x| \le 1]$

$\Rightarrow -1 \le \frac{4a-3}{7-a} \le 1 \Rightarrow a - 7 \le 4a - 3 \le 7 - a$

$\Rightarrow a \ge -4/3$ and $a \le 2$

Thus, $f'(x) = 0$ has solutions in R if $-4/3 \le a \le 2$

So, $f'(x) = 0$ is not solvable in R if $a < -4/3$ or $a > 2$, i.e.,

$a \in (-\infty, -4/3) \cup (2, \infty)$

202 **(b,d)**

$f(x) = \int_{-1}^{x} t(e^t - 1)(t-1)(t-2)^3(t-3)^5 dt$

$\Rightarrow f'(x) = x(e^x - 1)(x-1)(x-2)^3(x-3)^5$

The critical points are 0, 1, 2, 3

Sign scheme of $f'(x)$

- (−) 0 (−) 1 (+) 2 (−) 3 (+)

Clearly $x = 1$ and $x = 3$ are the points of minima

203 **(a,b,c,d)**

$f(x) = x^4(12\log_e x - 7); x > 0$

$\Rightarrow \frac{dy}{dx} = 16x^3(3\log_e x - 1)$ and $\frac{d^2y}{dx^2} = x^2(9\log_e x)$

$\frac{dy}{dx} = 0 \Rightarrow x = e^{1/3}$

At $x = e^{1/3}, \frac{d^2y}{dx^2} > 0$ hence $x = e^{1/3}$ is point of minima

Also, for $0 < x < 1, \frac{d^2y}{dx^2} < 0$ and for $x > 1, \frac{d^2y}{dx^2} > 0$

Hence $x = 1$ point of inflection and for $0 < x < 1$, graph is concave downward and for $x > 1$, graph is concave upward

204 **(a,b,c,d)**

$f(x) = x^{1/3}(x-1)$

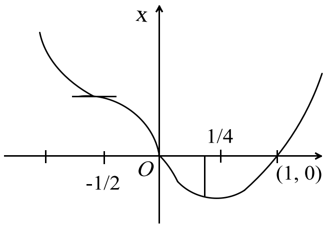
$\Rightarrow \frac{df(x)}{dx} = \frac{4}{3}x^{1/3} - \frac{1}{3}\cdot\frac{1}{x^{2/3}} = \frac{1}{3x^{2/3}}[4x-1]$

$\Rightarrow f'(x)$ changes sign from –ve to +ve, at $x = 1/4$, which is point of minima

Also, $f'(x)$ does not exist at $x = 0$ as $f(x)$ has vertical tangent at $x = 0$

$$f''(x) = \frac{4}{9}\cdot\frac{1}{x^{2/3}} + \frac{1}{3}\cdot\frac{2}{3}\cdot\frac{1}{x^{5/3}} = \frac{2}{9x^{2/3}}\left[2 + \frac{1}{x}\right]$$
$$= \frac{2}{9x^{2/3}}\left[\frac{2x+1}{x}\right]$$
$\therefore f''(x) = 0$ at $x = -\frac{1}{2}$ which is the point of inflection at $x = 0, f''(x)$ does not exists but $f''(x)$ changes sign, hence $x = 0$ is also the point of inflection

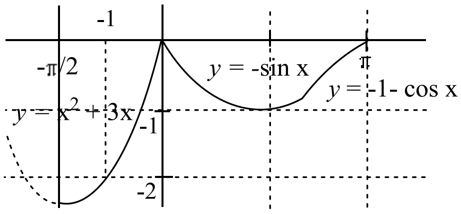
From the above information the graph of $y = f(x)$ is as shown



Also, minimum value of $f(x)$ is at $x = 1/4$ which is $-3 \times 2^{-8/3}$

Hence, range is $[-3 \times 2^{-8/3}, \infty)$

205 **(a,b,c,d)**



From the graph global minimum value is $f(-1) = -2$ and global maximum value is $f(0) = f(\pi) = 0$

206 **(c,d)**

r must be an even integer because two decreasing functions are required to make it increasing function

Let $y = r(n - r)$

When n is odd, $r = \frac{n-1}{2}$ or $\frac{n+1}{2}$ for maximum values of y when n is even, $r = \frac{n}{2}$ for maximum value of y

Therefore, maximum $(y) = \frac{n^2-1}{4}$ when n is odd and $\frac{\pi^2}{4}$ when n is even

207 **(a,b,c,d)**

a. $y^2 = 4ax \Rightarrow m_1 = y' = \frac{2a}{y}$

$$y = e^{-x}/2a \Rightarrow m_2 = y' = -\frac{1}{2a}e^{-x/2a} = -\frac{1}{2a}y$$

$m_1 m_2 = -1$. Hence, orthogonal

b. $y^2 = 4ax$

$\Rightarrow y' = \frac{4a}{2y_1} = \frac{2a}{y_1}$, not defined at $(0, 0)$

$x^2 = 4ay$

$\Rightarrow y' = \frac{2x_1}{4a} = \frac{x_1}{2a} = 0$ at $(0, 0)$

$\therefore$ The two curves are orthogonal at $(0, 0)$

c. $xy = a^2, x^2 - y^2 = b^2$

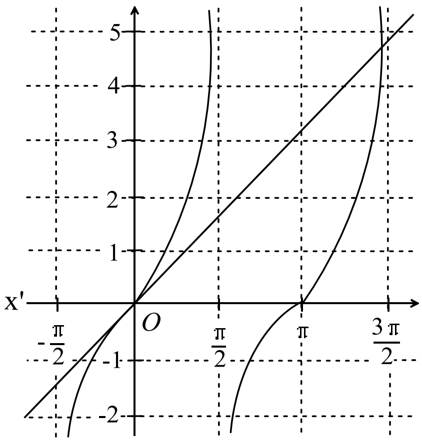
$$m_1 m_2 = -\frac{a^2}{x_1 y_1} = -\frac{a^2}{a^2} = -1 \Rightarrow \text{ orthogonal}$$

d. $y = ax, \Rightarrow y' = a$

$$x^2 + y^2 = c^2 \Rightarrow y' = -\frac{x_1}{y_1}$$

$m_1 m_2 = -\frac{ax_1}{y_1} = -\frac{y_1}{y_1} = -1 \Rightarrow$ orthogonal

208 **(c)**



It is clear from the graph that the curves $y = \tan x$ and $y = x$ intersect at P in $(\pi, 3\pi/2)$

Thus, the smallest +ve roots of $\tan x - x = 0$ is $(\pi, 3\pi/2)$

209 **(d)**

$$f(x) = \frac{x^2-1}{x^2+1} = \frac{(x^2+1)-2}{x^2+1} = 1 - \frac{2}{x^2+1}$$

For $f(x)$ to be min $\frac{2}{x^2+1}$ should be max, which is so if $x^2 + 1$ is min and $x^2 + 1$ is min at $x = 0$

$$\therefore f_{\min} = \frac{0-1}{0+1} = -1$$

210 **(a,b,d)**

$$y = \frac{2x-1}{x-2}$$
$$\frac{dy}{dx} = \frac{2(x-2)-(2x-1)}{(x-2)^2} = \frac{-3}{(x-2)^2} < 0 \ \forall x \neq 2$$

Therefore, y is decreasing in $(-\infty, 2)$ as well as in $(2, \infty)$

$$y = \frac{2x-1}{x-2} \Rightarrow x = \frac{2y-1}{y-2}$$

$\therefore f^{-1}(x) = \frac{2x-1}{x-2}$ $\therefore f(x)$ is its own inverse

211 **(a,c)**

$$f(x) = \frac{2-x}{\pi}\cos\pi(x+3) + \frac{1}{\pi^2}\sin\pi(x+3)$$
$$f'(x) = -\frac{1}{\pi}\cos\pi(x+3) - (2-x)\sin\pi(x+3)$$
$$+\frac{1}{\pi}\cos\pi(x+3) = (x-2)\sin\pi(x+3) = 0$$

$x = 2, 1, 3$

$f''(x) = \sin\pi(x+3) + \pi(x-2)\cos\pi(x+3)$

$f''(1) = -\pi < 0, f''(2) = 0, f''(3) = \pi > 0$

Therefore, $x = 1$ is a maximum and $x = 3$ is a minimum, hence $x = 2$ is the point of inflection

212 **(a,c,d)**

$\because\ f'(x) = (x^2-1)^n\{(x^2-1)(2x+1) + 2nx(x^2+x+1)\}$

$= (x+1)^{n+1}(x-1)^{n+1}(2x+1) + 2nx(x+1)^n(x-1)^n(x^2+x+1)$

If power even, them neither max not min.

$\therefore\ n = 2, 4, 6\quad (\because\ n+1 = \text{odd})$

213 **(a,b,c)**

$f(x) = \log(2x - x^2) + \sin\frac{\pi x}{2}$

$= \log(1-(x-1)^2) + \sin\frac{\pi x}{2}$

$f(1-x) = \log(1-(1-(x-1)^2)) + \sin\frac{\pi(1-x)}{2}$

$= \log(1-x^2) + \cos\frac{\pi x}{2}$

Also, $f(1+x) = \log(1-(1+(x-1)^2)) + \sin\frac{\pi(1+x)}{2}$

$= \log(1-x^2) + \cos x\frac{\pi x}{2}$

Hence, function is symmetrical about line $x = 1$

Also, $f(1) = 1$

Also, for domain of the function is $2x - x^2 > 0$ or $x \in (0, 2)$

For $x > 1, f(x)$ decreases hence $x = 1$ is point of maxima

Also, maximum value of the function is 1

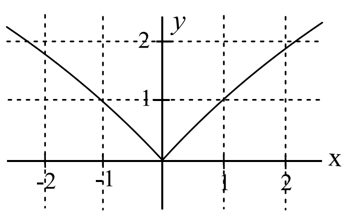
Also, $f(x) \to \infty$, when $x \to 2$, hence absolute minimum value of f does not exists

214 **(c,d)**

$f(x) = x^{6/7}$

$\Rightarrow f'(x) = -\frac{6}{7}\frac{13}{7}x^{\frac{-8}{7}}$, here $f'(x)$ does not change sign, hence has no point of inflection

Graph of $f(x) = x^{6/7}$



For $f(x) = x^6, f'(x) = 30x^4$, but $f'(x)$ does not changes sign in the neighbourhood of $x = 0$

$f(x) = \cos x + 2x$

$\Rightarrow f'(x) = -\cos x$,

$\Rightarrow f'(0) = 0$ for $x = (2n+1)\pi/2, n \in Z$

Also, sign $f'(x)$ changes sign in the neighbourhood of $(2n+1)\pi/2$, hence function has infinite points of inflection

$f(x) = x|x| = \begin{cases} -x^2, x < 0 \\ x^2, x \geq 0 \end{cases}$

$\Rightarrow f'(x) = \begin{cases} -2, x < 0 \\ x^2, x > 0 \end{cases}$, here $f'(x)$ changes sign in the neighbourhood of $x = 0$, hence has point of inflection

215 **(b)**

The maximum value of $f(x) = \cos x + \cos(\sqrt{2}x)$ occurs when $\cos x = 1$ and $\cos(\sqrt{2}x) = 1$

$\Rightarrow x = 2n\pi \in Z$ and $\sqrt{2}x = 2m\pi, m \in Z$

Comparing the value of $x, 2n\pi = \frac{2m\pi}{\sqrt{2}} \Rightarrow m = n = 0 \Rightarrow x = 0$ only

216 **(a,c)**

Given, $y = x^4$

$\Rightarrow\ \frac{dy}{dx} = 4x^3$

Equation of Tangent at (x, y) is

$Y - y = 4x^3(X - x)$

It pass through (2, 0), then

$0 - y = 4x^3(2 - x)$

$\Rightarrow\ -x^4 = 4x^3(2-x)\quad (\because y = x^4)$

$\Rightarrow\ x^3(8 - 4x + x) = 0$

$\Rightarrow\ x = 0,\ x = \frac{8}{3}$

$\therefore\ y = 0,\qquad y = \left(\frac{8}{3}\right)^4 = \frac{4096}{81}$

Point of contact are (0, 0) and $\left(\frac{8}{3}, \frac{4096}{81}\right)$

Equations of tangents are

$y - 0 = 0\ \Rightarrow\ y = 0$ and $y - \frac{4096}{81} = 4\left(\frac{8}{3}\right)^3\left(x - \frac{8}{3}\right)$

$\Rightarrow\ y = 0$ and $y - \frac{4096}{81} = \frac{2048}{27}\left(x - \frac{8}{3}\right)$

217 **(a,d)**

Since $g(a) \neq 0$, therefore either $g(a) > 0$ or $g(a) < 0$. Let $g(a) > 0$. Since $g(x)$ is continuous at $x = a$, therefore there exists a neighbourhood of a in which $g(x) > 0$.

$\Rightarrow f'(x) > 0 \Rightarrow f(x)$ is increasing in the neighbourhood of a. Let $g(a) < 0$. Since $g(x)$ is continuous at $x = a$, therefore there exists a neighbourhood of a in which $g(x) < 0$

$\Rightarrow f'(x) < 0 \Rightarrow f(x)$ is decreasing in the neightbourhood of a

218 **(a,c,d)**

$y = x(c - x)\qquad (1)$

$y = x^2 + ax + b\qquad (2)$

Slope of (1) curve $= c - 2x$

And at (1, 0), $c - 2 = m_1$ (say)
Slope of (2) curve $= 2x + a$
at $(1, 0), 2 + a = m_1$ (say)
Curves are touching at (1, 0)
$\Rightarrow m_1 = m_2$
$\Rightarrow 2 + a = c - 2$ (3)
Also (1, 0) lies on both the curves
$\Rightarrow 0 = c - 1$ and $0 = 1 + a + b$ (4)
Solving (3) and (4), we get
$a = -3, b = 2, c = 1$

219 **(a,c)**
$xy = (a + x)^2$
$\Rightarrow y + xy' = 2(a + x)$
Now $y' = \pm 1$
$\Rightarrow y \pm x = 2(a + x)$
$$\frac{(a+x)^2}{x} \pm x = 2(a + x)$$
$$\Rightarrow \pm x = 2(a + x) - \frac{(a+x)^2}{x}$$
$\Rightarrow \pm x^2 = (a + x)(x - a)$
$\Rightarrow \pm x^2 = x^2 - a^2$
$\Rightarrow 2x^2 = a^2 \Rightarrow x = \pm \frac{a}{\sqrt{2}}$

220 **(a,b)**
$\because F(x) = \int_0^x 2|t|\, dt$
$\therefore F'(x)2|x| = 1$
$\Rightarrow x = \pm \frac{1}{2}$
$\Rightarrow F\left(\frac{1}{2}\right) = \int_0^{1/2} 2|t|\, dt = \{t^2\}_0^{\frac{1}{2}} = \frac{1}{4}$
and $F\left(-\frac{1}{.2}\right) = \int_0^{1/2} 2|t|\, dt$
$= -2\int_0^{-1/2} t\, dt = -\frac{1}{4}$
$\therefore$ Equation of tangent at $\left(\frac{1}{2}, \frac{1}{4}\right)$ and at $\left(-\frac{1}{2}, -\frac{1}{4}\right)$ are
$y - \frac{1}{4} = 1.\left(x - \frac{1}{2}\right)$
$\Rightarrow y = x - \frac{1}{4}$
and $y + \frac{1}{4} = 1\left(x + \frac{1}{2}\right)$
$\Rightarrow y = x + \frac{1}{4}$

221 **(b,c,d)**
Since f is defined on $(0, \infty)$
Therefore, $2a^2 + a + 1 > 0$ which is true as $D < 0$
Also $3a^2 - 4a + 1 > 0$
$(3a - 1)(a - 1) > 0 \Rightarrow a < 1/3$ or $a > 1$ (1)
As f is increasing hence
$f(2a^2 + a + 1) > f(3a^2 - 4a + 1)$
$\Rightarrow 2a^2 + a + 1 > 3a^2 - 4a + 1$
$\Rightarrow 0 > a^2 - 5a$
$\Rightarrow a(a - 5) < 0 \Rightarrow (0, 5)$ (2)
From (1) and (2), we get
Hence, $a \in (0, 1/3) \cup (1,5)$
Therefore, possible integers are $\{2, 3, 4\}$

222 **(a,b)**
Since the intercepts are equal in magnitude but opposite in sign
$\Rightarrow \left.\frac{dy}{dx}\right|_P = 1$
Now $\frac{dy}{dx} = x^2 - 5x + 7 = 1$
$\Rightarrow x^2 - 5x + 6 = 0$
$\Rightarrow x = 2$ or 3

223 **(a,b,c)**
Clearly, $f(0) = 0$. So $f(x) = 0$ has two real roots $0, \alpha_0(> 0)$. Therefore, $f'(x) = 0$ has a real root α_1 lying between 0 and α_0. So, $0 < \alpha_1 < \alpha_0$
Again, $f'(x) = 0$ is a fourth-degree equation. As imaginary roots occur in conjugate pairs, $f'(x) = 0$ will have another real root α_2. Therefore, $f''(x) = 0$ will have a real root lying between α_1 and α_2. As $f(x) = 0$ is an equation of the fifth degree, it will have at least three real roots and so $f'(x)$ will have at least two real roots

224 **(a,b,d)**
$$f(x) = \frac{x}{1 - x^2}$$
$\therefore f'(x) = \frac{1 + x^2}{(1 - x^2)^2} = 1$, i.e., $x = 0, -\sqrt{3}, \sqrt{3}$
$\Rightarrow$ The point are $(0, 0), \left(\pm\sqrt{3}, \mp\frac{\sqrt{3}}{2}\right)$

225 **(a,b,c)**
a. Let $f(x) = e^x \cos x - 1$
$\Rightarrow f'(x) = e^x(\cos x - \sin x) = 0$
$\Rightarrow \tan x = 1$,which has a root between two roots of $f(x) = 0$
b. Let $f(x) = e^x \sin x - 1$,
$f'(x) = e^x(\sin x + \cos x) = 0$
$\Rightarrow \tan x = -1$, which has a root between two roots of $f(x) = 0$
c. Let $f(x) = e^{-x} - \cos x$,
$f'(x) = -e^{-x} + \sin x = 0$
$\Rightarrow e^{-x} = \sin x$, which has a root between two roots of $f(x) = 0$

226 **(a,b)**
$x^3 - y^2 = 0$ (1)
$\Rightarrow 2y \times \frac{dy}{dx} = 3x^2$
Slope of the tangent at

$P = \left.\frac{dy}{dx}\right|_P = \left.\frac{3x^2}{2y}\right|_{(4m^2,8m^3)} = 3m$

$\therefore$ Equation of the tangent at P is

$y - 8m^3 = 3m\,(x - 4m^2)$ or $y = 3mx - 4m^3$

(2)

It cuts the curve again at point Q. Solving (1) and (2), we get $x = 4m^2, m^2$

Put $x = m^2$ in equation (2)

$\Rightarrow y = 3m(m^2) - 4m^3 = -m^3 \;\therefore Q$ is $(m^2, -m^3)$

Slope of the tangent at $Q = \left.\frac{dy}{dx}\right|_{(m^2,-m^3)} =$

$\frac{3\,(m^4)}{2\times(-m^3)} = \frac{-3}{2}m$

Slope of the normal at $Q = \frac{1}{(-3/2)\,m} = \frac{2}{3m}$

Since tangent at P is normal at $Q \Rightarrow \frac{2}{3m} = 3m$

$\Rightarrow\; 9m^2 = 2$

227 **(a,b,d)**

f is not differentiable at $x = \frac{1}{2}$

g is not continuous in [0, 1] at $x = 0$

h is not continuous in [0, 1] at $x = 1$

$k(x) = (x+3)^{\text{In}_2 5} = (x+3)p$, where $2 < p < 3$, which is continuous and differentiable

228 **(a,b,c,d)**

a. $y^2 = 4ax$ and $y = e^{-x/2a}$

$y' = \frac{2a}{y}$ and $y' = -\frac{1}{2a}\,e^{-x/2a} = -\frac{1}{2a}y$

Let the intersection point be (x_1, y_1)

$y' = \frac{2a}{y_1}$ and $y' = -\frac{1}{2a}\,y_1$

$m_1 m_2 = -1$. Hence orthogonal

b. $y^2 = 4ax$ and $x^2 = 4ay$

$y' = \frac{4a}{2y_1} = \frac{2a}{y_1}$, not defined at $x = 0$

$y' = \frac{2x_1}{4a} = \frac{x_1}{2a}$ 0 at $x = 0$

$\therefore$ The two curves are orthogonal at (0, 0)

c. $xy = a^2$and $x^2 - y^2 = b^2$

$m_1 m_2 = -\frac{a^2}{x_1 y_1} = -\frac{a^2}{a^2} = -1$ orthogonal

d. $y = ax$ and $x^2 + y^2 = c^2$

$y' = a$ and $y' = -\frac{x_1}{y_1}$

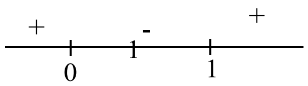
$m_1 m_2 = -\frac{ax_1}{y_1} = -\frac{y_1}{x_1} = -1$ orthogonal

229 **(a,b,c,d)**

$$f'(x) = 2 - 2x^{-1/3} = 2\left(1 - \frac{1}{x^{1/3}}\right)$$

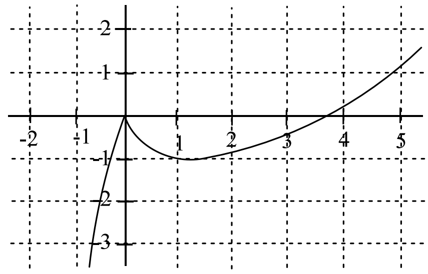
$$= 2\left(\frac{x^{1/3} - 1}{x^{1/3}}\right)$$

Sign scheme of derivative is



$f(x)$ has point of maxima at $x = 0$ and point of minima at $x = 1$

Also $f(x)$ is non-differentiable at $x = 0$



230 **(d)**

1. $f(x) = |x|$ are not differential at $x = 0$

2. $f(x) = \tan x$ is discontinuous at $x = \frac{\pi}{2}$

3. $f(x) = 1 + (x-2)^{\frac{2}{3}}$ gives real and imaginary value at $x = 1$

4. Only function which satisfies Rolle's therorem is

$f(x) = x(x-2)^2$ in $0 \le x \le 2$

231 **(a,b,c,d)**

$f(x) = 2x - \sin x \Rightarrow f'(x) = 2 - \cos x > 0\;\; \forall x$.

Hence, $f(x)$ is strictly increasing, hence one-one and onto $g(x) = x^{1/3}$

$\Rightarrow g'(x) = \frac{1}{3}x^{-2/3} > 0\; \forall x$, hence $g(x)$ is strictly increasing and hence one-one and onto

Also, gof is one-one

$gof(x) = (2x - \sin x)^{1/3}$ has range R as the range of $2x - \sin x$ is R

232 **(a,b,d)**

$f(x) = 2x^3 + 9x^2 + 12x + 1$

$\Rightarrow f'(x) = 6[x^2 + 3x + 2]$

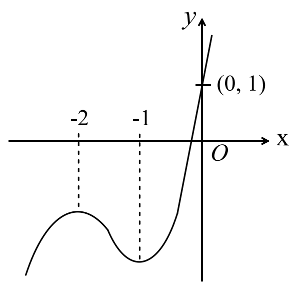
$= 6(x+2)(x+1)$

$f'(x) < 0$ for $x \in (-2, -1)$, where $f(x)$ decreases

$f'(x) > 0$ for $x \in (-\infty, -2) \cup (-1, \infty)$, where $f(x)$ increases

$f''(x) = 2x + 3 = 0$

$\Rightarrow x = -3/2$ is the point of inflection



From the graph, f is many-one, hence it is not bijective

233 **(a,b,c)**

$$f(x) = \frac{\sin(x+a)}{\sin(x+b)}$$

$f'(x)$

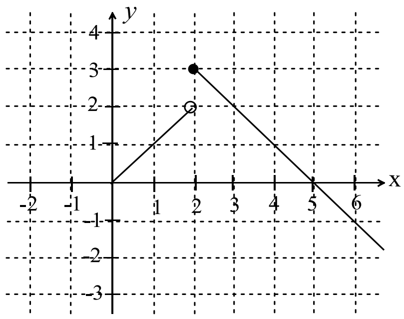
$$= \frac{\sin(x+b)\cos(x+a) - \sin(x+a)\cos(x+b)}{\sin^2(x+b)}$$

$$= \frac{\sin(b-a)}{\sin^2(x+b)}$$

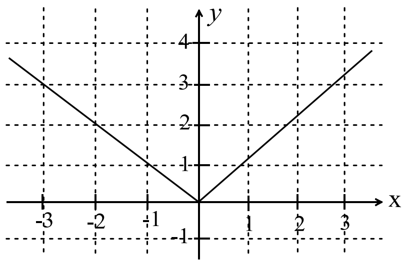
If $\sin(b-a) = 0$, then $f'(x) = 0 \Rightarrow f(x)$ will be a constant, i.e., $b - a = n\pi$ or $b - a = (2n+1)\pi$ or $b - a = 2n\pi$, then $f(x)$ has no minima

234 **(a,b,c)**

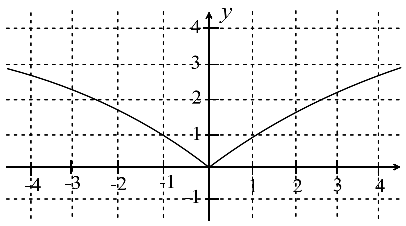
The following function is discontinuous at $x = 2$, but has point of maxima



$f(x) = |x|$ has point of minima at $x = 0$, through it is non-differentiable at $x = 0$



$f(x) = x^{2/3}$ has point of inflection at $x = 0$, as curve changes its concavity at $x = 0$, however $x = 0$ is point of minima for the function



235 **(a,c)**

$f'(x) = 4(x^3 - 3x^2 + 3x - 1) = 4(x-1)^3 > 0$ for $x > 1$

Hence, f increases in $[1, \infty)$. Moreover, $f'(x) < 0$ for $x < 1$

Hence, f has a minimum at $x = 1$

236 **(a,c)**

Given, $h(x) = f(x) - (f(x))^2 + (f(x))^3$

$\therefore\ h'(x) = f'(x) - 2f(x)f'(x) + 3(f(x))^2 f'(x)$

$= f'(x)\{1 - 2f(x) + 3(f(x))^2\}$

Discriminant of the quadratic equation

$3(f(x))^2 - 2f(x) + 1 = 0$

Is $(-2)^2 - 4.3.1 = -8 < 0$

$\Rightarrow 3(f(x))^2 - 2f(x) + 1 > 0, \forall f(x) \in R (\because 3 > 0)$

$\Rightarrow$ Sign of $h'(x)$ will be same as that of sigh of $f'(x)$

$\therefore$ $h(x)$ is increasing whenever $f(x)$ is increasing and $h(x)$ is decreasing whenever $f(x)$is decreasing.

237 **(b,d)**

For $y^2 = 4\,ax$, y-axis is tangent at (0, 0), while for $x^2 = 4ay$, x-axis is tangent at (0, 0)

Thus the two curves cut each other at right angles

$\therefore$ Also for $y^2 = 4ax, \frac{dy}{dx} = \frac{2a}{y} = m_1$

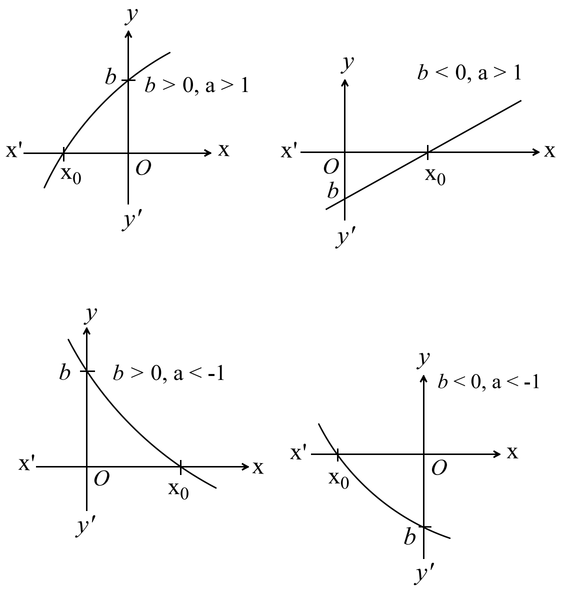
For $y = e^{-x/2a}, \frac{dy}{dx} = \frac{-1}{2a}\, e^{-x/2a} = \frac{-y}{2a} = m_2$

$\Rightarrow m_1 m_2 = -1$

$\Rightarrow y^2 = 4ax$ and $y = e^{-x/2a}$ intersect at right angle

239 **(a,b,c)**

$f'(x) = \cos x + a$



If $a > 1$, then $f'(x) > 0$ or $f(x)$ is an increasing function, then $f(x) = 0$ has +ve root if $b < 0$ and –ve root if $b > 0$

$f'(x) = \cos x + a$

If $a < -1$, then $f'(x) < 0$ or $f(x)$ is a decreasing function, then $f(x) = 0$ has negative root if $b < 0$

240 **(d)**

$$f'(x) = \frac{2(ad - bc)}{(ce^x + de^{-x})^2}$$

and $f(x)$ is an increasing function

$\therefore\ f'(x) > 0$

$\Rightarrow \dfrac{2(ad-bc)}{(ce^x+de^{-x})^2} > 0$

$\therefore\ 2(ad-dc) > 0$

$\Rightarrow\ ad > bc$

$\Rightarrow\ bc < ad$

241 **(a)**

Statement 2 is obviously true

Also, for $f(x) = 2\cos x + 3\sin x = \sqrt{13}\sin\Big(x +$ tan-123

$\Rightarrow$ g$(x) = \sin^{-1}\frac{x}{\sqrt{13}} - \tan^{-1}\frac{2}{3}$. Hence, statement 1 is true

242 **(a)**

Consider, $F(x) = e^{-\lambda x} f(x), \lambda \in R$

$F(0) = f(0) = 0$

$F(1) = e^{-1} f(1) = 0$

$\therefore$ By Rolle's theorem, $F'(c) = 0$

$F'(x) = e^{-\lambda x}\left(f'(x) - \lambda f(x)\right)$

$F'(c) = 0 \Rightarrow e^{-\lambda c}\left(f'(c) - \lambda f(c)\right) = 0$

$\Rightarrow f'(c) = \lambda f(c), 0 < c < 1$

243 **(a)**

Given, $x^2 + y^2 = 25$

$\Rightarrow\ 2x\,dy + 2y\,dy = 0$

$\Rightarrow \dfrac{dy}{dx} = -\dfrac{x}{y}$

$\Rightarrow\ \dfrac{dy/dt}{dx/dt} = -\dfrac{x}{y}$

$\Rightarrow\ -\dfrac{1.5}{dx/dt} = -\dfrac{3}{4}$

$\Rightarrow\ \dfrac{dx}{dt} = \dfrac{1.5\times 4}{3} = 2$ cm/s

244 **(c)**

$f(x) = \dfrac{1}{e^x\frac{2}{e^x}}$

Using $AM \geq GM, \frac{e^x+\frac{2}{e^x}}{2} \geq \left(e^x.\frac{2}{e^x}\right)^{\frac{1}{2}}, as\ e^x > 0$

$\Rightarrow\ e^x + \dfrac{2}{e^x} \geq 2\sqrt{2} \Rightarrow 0 < \dfrac{1}{e^x+\frac{2}{e^x}} \leq \dfrac{1}{2\sqrt{2}}$

$\therefore\ 0 < f(x) \leq \frac{1}{2\sqrt{2}}$, for all $x \in R$

$\Rightarrow$statement II is true and statement I as for some 'e'

$\Rightarrow\ f(c) = \frac{1}{3'}$ which is not true

Alternate

$f(x) = \dfrac{1}{e^x + 2e^{-x}} = \dfrac{e^x}{e^{2x}+2}$

$\Rightarrow f'(x) = \dfrac{(e^{2x}+2)e^x - 2e^{2x}.e^x}{(e^{2x}+2)^2}$

$\Rightarrow f'(x) = 0 \Rightarrow e^{2x} + 2 = 2e^{2x}$

$\Rightarrow\ e^{2x} = 2 \Rightarrow e^x = \sqrt{2}$

Maximum value of $f(x) = \frac{\sqrt{2}}{4} = \frac{1}{2\sqrt{2}}$

245 **(d)**

Though $|x-1|$ is non-differentiable at $x = 1, (x-\ 1)|x-1|$ is differentiable at $x = 1$, for which Lagrange's mean value theorem is applicable

246 **(a)**

$\frac{dy}{dx} = 12x\,(x^2 - x + 1) + a$ and $\frac{d^2y}{dx^2} =$ $12(3x^2 - 2x + 1) > 0$

$\Rightarrow \frac{dy}{dx}$ is an increasing function

But $\frac{dy}{dx}$ is a polynomial of degree 3 $\Rightarrow$ it has exactly one real root

247 **(b)**

$f'(x) = 6x^2 - 18x + 12$

For increasing function, $f'(x) > 0$

$\therefore 6(x^2 - 3x + 2) > 0$

$\Rightarrow 6(x-2)(x-1) > 0$

$\Rightarrow x < 1\ and\ x > 2$

$\therefore f(x)$Is increasing outside the interval (1, 2), therefore it is true statement.

Now, $f'(x) < 0$

$\Rightarrow 6(x-2)(x-1) < 0$

$\Rightarrow 1 < x < 2$

$\therefore$A and R are both true, but, R is not the correct reason.

248 **(d)**

Given, $y^2 = x + \sin x$

$\Rightarrow \quad 2y\dfrac{dy}{dx} = 1 + \cos x$

Here, $\frac{dy}{dx} = 0$

$\Rightarrow \quad \cos x = -1$

$\Rightarrow \ \sin x = 0$

$\therefore$ From Eq. (i), $y^2 = x$

$\because \quad f'(-1) = f'\left(\dfrac{1}{3}\right) = 0$

$\therefore \quad f'(x) = a(x+1)\left(x - \dfrac{1}{3}\right)$

$= a'\left(x^2 + \dfrac{2}{3}x - \dfrac{1}{3}\right)$

On integrating w.r.t. x, we get

$f(x) = \frac{a'}{3}(x^3 + x^2 - x) + \lambda \quad$ ...(i)

Where λ is constant of integration

And $\ f(-2) = 0$

$\Rightarrow \quad \dfrac{a'}{3}(-8 + 4 + 2) + \lambda = 0$

$\therefore \ \lambda = \dfrac{2a'}{3}$

From Eq. (i),

$f(x) = \dfrac{a'}{3}(x^3 + x^2 - x + 2)$

Also, $\int_{-1}^{1} f(x)\,dx = \frac{14}{3}$

$\Rightarrow \quad \dfrac{a'}{3}\displaystyle\int_{-1}^{1}(x^3 + x^2 - x + 2)\,dx = \dfrac{14}{3}$

$\Rightarrow \quad 0 + \dfrac{2a'}{3}\displaystyle\int_{0}^{1}(x^2 - 2)\,dx = \dfrac{14}{3}$

$\Rightarrow \quad \dfrac{2a'}{3}\left\{\dfrac{1}{3} + 2\right\} = \dfrac{14}{3}$

$\Rightarrow \quad \dfrac{14a'}{9} = \dfrac{14}{3}$

$\Rightarrow \quad a' = 3$

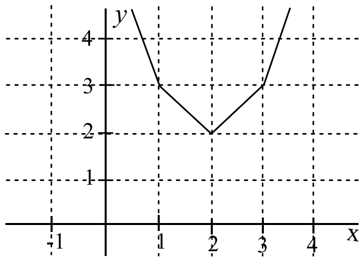
Then, $f(x) = x^3 - x^2 - x + 2$

On comparing with $f(x) = ax^3 + bx^2 + cx + d$

$\therefore \ a = 1, b = 1, c = -1, d = 2$

249 **(d)**

Statement 2 is true as $f(x)$ is non-differentiable at $x = 1, 2, 3$. But $f(x)$ has a point of minima at $x = 1$ and not at $x = 3$



250 **(b)**

$x(x+3)$ and $e^{-x/2}$ are continuous and differentiable every where, so $x(x+3)e^{-x/2}$ is continuous and differentiable

And $\ f(-3) = f(0) = 0$

And $\ f'(x) = (x^2 + 3x).\,e^{-x/2}$

$= \left(-\dfrac{1}{2}\right) + e^{-x/2}.\,(2x + 3)$

$= -\dfrac{1}{2}e^{-x/2}(x^2 + 3x - 4x - 6)$

$= -\dfrac{1}{2}e^{-x/2}(x^2 - x - 6)$

$= -\dfrac{1}{2}e^{-x/2}(x - 3)(x + 2)$

$\therefore \quad f'(x) = 0 \ \Rightarrow \ x = 3, -2$

$3 \notin [-3, 0]$

$\therefore \quad x = -2 \in [-3, 0]$

$\therefore$ Rolle's theorem is verified

LMVT is also applied

$\because \quad f'(c) = \dfrac{f(b) - f(a)}{b - a}$

ie, Rolle's theorem is a special case of LMVT

Since, $f(a) = f(b) \ \Rightarrow \ f'(c) = 0$

251 **(a)**

$f(x) = \frac{x^3}{3} + \frac{ax^2}{2} + x + 5$

$\Rightarrow f'(x) = x^2 + ax + 1$

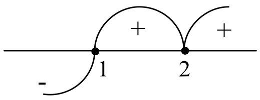
If $f(x)$ has positive point of maxima, then point of minima is also positive. Hence, both the roots of equation $x^2 + ax + 1 = 0$ must be positive

$\Rightarrow$ sum of roots $-a > 0$, product of roots $1 > 0$ and discriminant $D = a^2 - 4 > 0$

$\Rightarrow a < -2$

252 **(c)**
It is clear from figure $f''(x)$ has no sign change at $x = 2$

Hence, $f(x)$ is neither maximum nor minimum at $x = 2$



253 **(a)**
Let $y = \sqrt{-3 + 4x - x^2}$

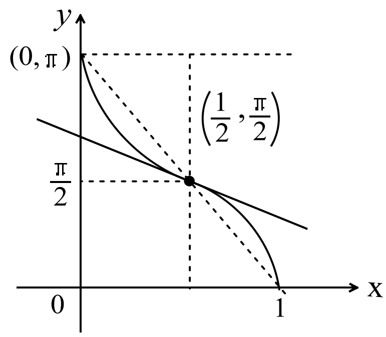
$\Rightarrow x^2 + y^2 - 4x + 3 = 0$ or point (x, y) lies on this circle

Then, the given expression is $(y + 4)^2 + (x - 5)^2$, which is the square of distance between point $P(5, -4)$ and any point on the circle $x^2 + y^2 - 4x + 3 = 0$ which has centre $C(2, 0)$ and radius 1

Now $CP = 5$, then the maximum distance between the point P and any point on the circles is 6

$\Rightarrow$ Maximum value of $\left(\sqrt{-3 + 4x - x^2} + 4\right)^2 + (x - 5)^2$ is 36

254 **(b)**
Point of inflection of the curve is $\left(\frac{1}{2}, \frac{\pi}{2}\right)$ and this satisfies the line L



Slope of the tangent to the curve C at $\left(\frac{1}{2}, \frac{\pi}{2}\right)$

$$\frac{dy}{dx} = \frac{-2}{\sqrt{1 - (2x - 1)^2}} = \frac{-1}{\sqrt{x - x^2}} = -(x - x^2)^{-1/2}$$

$$\frac{d^2y}{dx^2} = \frac{1}{2}\frac{(1 - 2x)}{(x - x^2)^{3/2}} = 0; x = \frac{1}{2}$$

$$\left.\frac{dy}{dx}\right|_{x=1/2} = -2$$

As the slope decreases from -2, line cuts the curve at three distinct points and the minimum slope of the line when it intersects the curve at three distinct points is

$$\frac{\pi - \frac{\pi}{2}}{0 - \frac{1}{2}} = -\pi$$

$\therefore \frac{p}{2} \in [-\pi, -2] \Rightarrow p \in [-2\pi, -4]$

255 **(b)**
Let $h(x) = f(x) - f(a) + \lambda(x^3 - a^3)$

Where, λ is selected in such a way

$h(b) = f(b) - f(a) + \lambda(b^3 - a^3) = 0$ ...(i)

but $h(a) = 0$

hence, $h(x)$ satisfies all conditions of Rolle's theorem

$\because c \in (a, b)$

$\therefore h'(c) = 0$

$\Rightarrow f'(c) - 0 + 3\lambda c^2 = 0$

$\Rightarrow \lambda = \frac{\phi'(\chi)}{3\chi^2}$

From Eq. (i), $\lambda = -\frac{f(b)-f(a)}{(b^3-a^3)}$

$$\therefore \frac{f(b)-f(a)}{(b^3-a^3)} = \frac{f'(c)}{3c^2}$$

256 **(c)**
$f(x) = (x^3 - 6x^2 + 12 - 8)e^x$

$$\Rightarrow f'(x) = e^x(x^3 - 6x^2 + 12x - 8) + e^x(3x^2 - 12x + 12)$$

$$= e^x(x^3 - 3x^2 + 4)$$

$$\Rightarrow f'(x) = e^x(x^3 - 3x^2 + 4) + e^x(3x^2 - 6x)$$

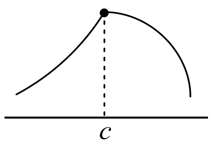
$$= e^x(x^3 - 6x + 4)$$

$$\Rightarrow f'(x) = e^x(x^3 - 6x + 4) + e^x(3x^2 - 6)$$

$$= e^x(x^3 + 3x^2 - 6x - 2)$$

Clearly, $f'(2) = f'(2) = 0$ and $f' \neq 0$, hence $x = 2$ is the point of inflection and hence not a point of extrema. Thus, statement 1 is true

But statement 2 is false, as it is not necessary that at point of inflection, extrema does not occur. Consider the following graph (figure)



S-2

257 **(a)**
$f'(x) = \text{In}\left(x + \sqrt{1+x^2}\right) = -\text{In}\left(\sqrt{1+x^2} - x\right)$

$\Rightarrow f'(x) > 0$ for $x > 0$ and $f'(x) < 0$ for $x < 0$

$\Rightarrow f(x)$ is increasing when $x > 0$ and decreasing for $x < 0$

Hence, for $x > 0, f(x) > f(0) \Rightarrow f(x) > 0$

Again $f(x)$ is decreasing in $(-\infty, 0)$

Then for $x < 0, f(x) > f(0) \Rightarrow f(x) > 0$

$\Rightarrow f(x)$ is positive for all $x \in R_0$

Thus, Statement 1 is true and follows from Statement 2

258 **(c)**
Statement 1 is true, but statement 2 is false as consider the functions in statement 1 in $\left(0, \frac{\pi}{2}\right)$

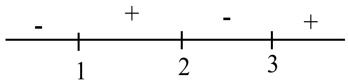
259 **(a)**
$f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$

$$f'(x) = 4x^3 - 24x^2 + 44x - 24$$

$$= 4(x^3 - 6x^2 + 11x - 6)$$

$$= 4(x-1)(x-2)(x-3)$$

Sign scheme of $f'(x)$



From the sign scheme of $f'(x), f(x)$ increases for $x \in (1,2) \cup (3, \infty)$

Since $f(x)$ is a polynomial function, which is continuous, and has no point of inflection, intervals of increase and decrease occur alternatively

260 **(b)**
$f(x) = x + \cos x$

$\therefore f'(x) = 1 - \sin x > 0 \; \forall x \in R,$

$\therefore f(x)$ is increasing

Statement 2 is true but does not explain statement 1

Therefore, according to statement 2, $f'(x)$ may vanish at finite number of points but in statement 1 $f'(x)$ vanishes at infinite number of points

261 **(c)**
Statement 1 is correct is $f(-2) = f(2) = 0$ and Rolle's theorem is not applicable, then it implies that either $f(x)$ is discontinuous or $f'(x)$ does not exist at at least one point in $(-1, 1)$. Since it is given that $\text{g}(x)$ is differentiable, $\text{g}(x) = 0$ has at least one value of x in $(-1, 1)$

Statement 2 is false as $f(x)$ must be differentiable in (a, b) is not given

262 **(d)**
When $x = 1, y = 1$

$$\frac{dy}{dx} = 3x^2 - 2x - 1 \Rightarrow \left(\frac{dy}{dx}\right)_{x=1} = 0$$

$\Rightarrow$ Equation of the tangent is $y = 1$

Solving with the curve, $x^3 - x^2 - x + 2 = 1$

$\Rightarrow\ x^3 - x^2 - x + 1 = 0\ \Rightarrow\ x = -1, 1$ (1 is repeated root)

$\therefore$ the tangent meets the curve again at $x = -1$

$\therefore$ statement 1 is false and statement 2 is true

263 **(a)**
Suppose $f(x) = 0$ has real root say $x = a$, then $f(x) < 0$ for all $x < a$

Thus $|f(x)|$ becomes strictly decreasing in $(-\infty, a)$ which is a contradiction

264 **(c)**
Both $f(x) = x$ and $g(x) = x^3$ are increasing in $(-1, 0)$ but $h(x) = x, x^3$ is decreasing

265 **(c)**
Statement 1 is correct as it is the statement of Cauchy's mean value theorem. Statement 2 is false as it is necessary that c in both $f'(c) = \frac{f(b) - f(a)}{b-a}$ and $g'(c) = \frac{g(b) - g(a)}{b-a}$ is same

266 **(b)**
$f(x) = (x-1)(x-2)(x-3)$

$= x^3 - 6x^2 + 11x - 6$

$\therefore\quad f'(x) = 3x^2 - 12x + 11$

Put $f'(x) = 0$

$\Rightarrow\quad 3x^2 - 12x + 11 = 0$

$$\Rightarrow\quad x = \frac{12 \pm \sqrt{144 - 132}}{6} = 2 \pm \frac{1}{\sqrt{3}}$$

$$\therefore\quad 1 \leq 2 \pm \frac{1}{\sqrt{3}} \leq 3$$

267 **(b)**
$f(x) = \sin(\cos x)$

$\Rightarrow f'(x) = -\sin x \cos(\cos x) < 0$ for $\forall x \in \left[0, \frac{\pi}{2}\right]$

Statement 2 is also true, but it is not the only reason for statement 1 to be correct

268 **(a)**
Given $f(x) = xe^{-x}$

$f'(x) = e^{-x} - xe^{-x}$
$f''(x) = -e^{-x} - e^{-x} + xe^{-x}$
$= -2e^{-x} + xe^{-x}$
For maximum, put $f'(x) = 0 \quad \Rightarrow \quad x = 1$
And $\quad f''(1) = -1 < 0$
$\therefore$ Both A and R are true and R is the correct reason for A.

269 **(a)**

$$f(x) = \frac{\log_e x}{x} \Rightarrow f'(x) = \frac{1 - \log_e x}{x^2}$$

$f'(x) > 0$ for $1 - \log_e x > 0$ or $x < e \Rightarrow f(x)$ is increasing

$f(x)$ is decreasing for $x > e$

$e < 2.91 < \alpha < \beta$

$\Rightarrow f(\alpha) > f(\beta)$

$$\Rightarrow \frac{\log_e \alpha}{\alpha} > \frac{\log_e \beta}{\beta}$$

$\Rightarrow \beta \log_e \alpha > \alpha \log_e \beta$

$\Rightarrow \alpha^\beta > \beta^\alpha$

270 **(a)**
Equation of a tangent at (h, k) on $y = f(x)$ is

$y - k = f'(h)(x - h)$ (1)

Suppose (1) passes through (a, b)

$b - k = f'(h)[a - h]$ must hold good for some (h, k)

Now $hf'(h) = f(h) - af'(h) + b = 0$ represents an equation of degree odd in h

$\therefore\ \exists$ some 'h' for which LHS vanishes

271 **(a)**
Consider $f'(x) = 4ax^3 + 3bx^2 + 2cx + d$

$\Rightarrow f(x) = ax^4 + bx^3 + cx^2 + dx + e$

$f(0) = e$ and $f(3) = 81a + 27b + 9c + 3d + e$

$= 3\,(27a + 9b + 3c + d) + e = e$

Hence, Rolle's theorem is applicable for $f(x)$

$\Rightarrow$ there exists at least one c in (a, b) such that $f'(c) = 0$

272 **(b)**

Let $f(x) = \sin x \tan x - x^2 \Rightarrow f'(x) =$
$\sin x \sec^2 x + \sin x - 2x$

$\Rightarrow f''(x) = 2\sin x \sec^2 x \tan x + \sec x + \cos x - 2$

$= 2\sin x \tan x \sec^2 x + (\cos x + \sec x - 2) > 0\ \forall x \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow f'(x)$ is an increasing function

$\Rightarrow f'(x) > f'(0) \Rightarrow \sin x \sec^2 x + \sin x - 2x > 0$

$\Rightarrow f(x)$ is an increasing function

$\Rightarrow f(x) > f(0)$

$\Rightarrow \sin x \tan x - x^2 > 0$

$\Rightarrow \sin x \tan x > x^2$

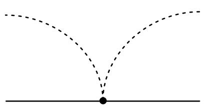
Thus, statement 1 is true, also statement 2 is true but it does not explain statement 1

273 **(a)**
Verify by taking $f(x) = lx^2 + mx + n$ in $[a, b]$

274 **(b)**
For Rolle's Theorem and LMVT, $f(x)$ must be continuous in $[a, b]$ and differentiable in (a, b)

Hence, Statement I is true

Since, $f(x) = |\sin|x||$ in $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ is non-differentiable in $x = 0$



Hence, Statement II is also true

275 **(a)**
For no extremum

$\frac{dy}{dx} > 0$ or $\frac{dy}{dx} < 0$ for all $x \in R$

$\therefore \frac{dy}{dx} = 3x^2 + 2ax + b > 0$

$\Rightarrow 3x^2 + 2ax + b > 0$

$\therefore D < 0$

$\Rightarrow 4a^2 - 4.3.b < 0$

$\Rightarrow a^2 < 3b$

276 **(d)**
Statement 1 is false as $f(x) = 5 - 4(x-2)^{2/3}$ attains the greatest value at $x = 2$, through it is not differentiable at $x = 2$, and for extreme value if is not necessary that $f'(x)$ exists at that point

Statement 2 is obviously true

277 **(c)**
$f'(x) = 3x^2 + 2x + 3 + \cos x$

$= 3\left\{x^2 + \frac{2x}{3} + 1\right\} + \cos x$

$= 3\left\{\left(x + \frac{1}{3}\right)^2 - \frac{1}{9} + 1\right\} + \cos x$

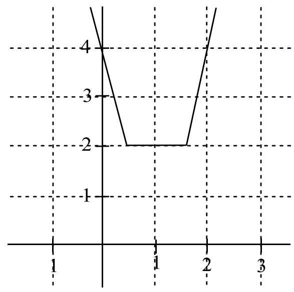
$= 3\left(x + \frac{1}{3}\right)^2 + \frac{8}{3} + \cos x > 0$

$\left[\because\ 3\left(x + \frac{1}{3}\right)^2 \geq 0, -1 \leq \cos x \leq 1\right]$

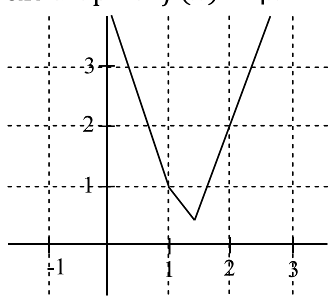
$\therefore f(x)$ is an increasing function

$\Rightarrow f(x)$ is one-one

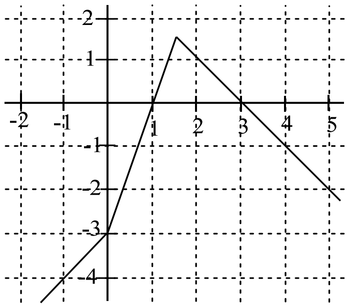
278 **(b)**
a.s. Graph of $f(x) = |2x - 1| + |2x - 3|$



From the graph $f(x)$ has infinite points of minima
b.s. $f(x) = 2\sin x - x \Rightarrow$ for $f'(x) = 2\cos x - 1 = 0$ we have $\cos x = 1/2$ which has infinite points of extrema
c.r. Graph of $f(x) = |x - 1| + |2x - 3|$



From the graph $f(x)$ has one points of minima
d.q. Graph of $f(x) = |x| - |2x - 3|$



From the graph $f(x)$ has one point of maxima

279 **(a)**

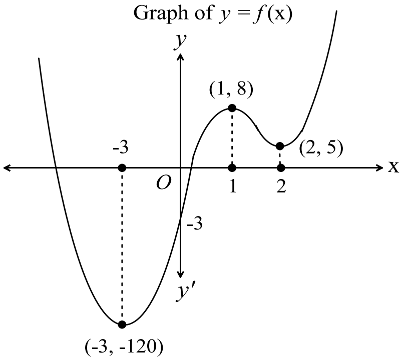
$f''(x) = 4x^3 - 28x + 24$

$= 4(x^3 - 7x + 6)$

$= 4(x^3 - x^2 + x^2 - x - 6x + 6)$

$= 4(x-1)(x^2 + x - 6)$

$= 4(x-1)(x+3)(x-2)$



Now, nature of roots of $f(x) + p = 0$ can be obtained by shifting the graph of $y = f(x)$ by p units upwards or downward depending on whether p is positive or negative

280 **(a)**

a. Given $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$ (say)

$\therefore da = 2R \cos A \, dA$

$db = 2R \cos B \, dB$

$dc = 2R \cos C \, dC$

$\therefore \frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 2R\,(dA + dB + dC)$ (1)

Also $A + B + C = \pi$ So, $dA + dB + dC = 0$ (2)

From equations (1) and (2), we get

$$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} + 1 = 1$$

$\Rightarrow m = \pm 1$

b. $x^2y^2 = 16 \Rightarrow xy = \pm 4$ (1)

$L_{ST} = \left|\frac{y}{dy/dx}\right|$

Differentiating (1) w.r.t x, we get $y + xy' = 0 \Rightarrow y' = \frac{-y}{x}$

$L_{ST} = \left|\frac{y}{y/x}\right| = |x| \Rightarrow L_{ST} = 2$

$\Rightarrow k = \pm 2$

c. $y = 2e^{2x}$ intersects y-axis at $(0, 2)$

$\frac{dy}{dx} = 4e^{2x}$ $\therefore \left.\frac{dy}{dx}\right|_{\text{at } x=0} = 4$

$\therefore$ Angle of intersection with y-axis $= \frac{\pi}{2} - \tan^{-1} 4 = \cot^{-1} 4$

$\Rightarrow n = 2$ or -1

d. $\frac{dy}{dx} = e^{\sin y} \cos y$: slope of the normal at $(1, 0) = -1$

equation of the normal is $x + y = 1$

Area $= \frac{1}{2}$

$\Rightarrow t = 1, -2$

283 **(b)**

a. $r = 6$ cm $\delta r = 0.06$

$A = \pi r^2 \Rightarrow \delta A = 2\pi r \delta r = 2\pi\,(6)(0.06) = 0.72\,\pi$

b. $V = x^3, \delta V = 3x^2 \delta x$

$$\frac{\delta V}{V} \times 100 = 3\frac{\delta x}{x} \times 100 = 3 \times 2 = 6$$

c. $(x-2)\frac{dx}{dt} = 3\frac{dx}{dt}$

$\Rightarrow x = 5$

d. $A = \frac{\sqrt{3}}{4} x^2$

$$\Rightarrow \frac{dA}{dt} = \frac{\sqrt{3}}{2}\left(x\frac{dx}{dt}\right) = \frac{\sqrt{3}}{2} \times 30 \times \frac{1}{10} = \frac{3\sqrt{3}}{2}$$

284 **(b)**

Given, $2y^2 = x + 1$

$$\Rightarrow \quad 4y\frac{dy}{dx} = 1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{4y}$$

$\therefore$ The slope of the normal is

$$\frac{dx}{dy} = -4y$$

5. $\left(\frac{dx}{dy}\right)_{(7,2)} = -4(2) = -8$

6. $\left(\frac{dx}{dy}\right)_{(0,\frac{1}{\sqrt{2}})} = \frac{-4}{\sqrt{2}} = -2\sqrt{2}$

7. $\left(\frac{dx}{dy}\right)_{(1,-1)} = -4(-1) = 4$

8. $\left(\frac{dx}{dy}\right)_{(3,\sqrt{2})} = -4(\sqrt{2}) = -4\sqrt{2}$

$\therefore$ Options (b) satisfy the all four statements.

285 **(c)**

a. $y^2 = 4x$ and $x^2 = 4y$ intersect at point (0, 0) and (4, 4)

$C_1 : y^2 = 4x$ $C_2 : x^2 = 4y$

$\frac{dy}{dx} = \frac{2}{y}$ $\frac{dy}{dx} = \frac{x}{2}$

$\left.\frac{dy}{dx}\right|_{0,0} = \infty$ $\left.\frac{dy}{dx}\right|_{0,0} = 0$

Hence, $\tan\theta = 90°$ at point (0, 0)

$\left.\frac{dy}{dx}\right|_{(4,4)} = \frac{1}{2}$ $\left.\frac{dy}{dx}\right|_{(4,4)} = 2$

$\tan\theta = \left|\frac{2-\frac{1}{2}}{1+2\times\frac{1}{2}}\right| = \frac{3}{4}$

b. Solving I : $2y^2 = x^3$ and II : $y^2 = 32x$; we get $(0,0), (8,16)$ and $(8,-16)$

at $(0,0)\left.\frac{dy}{dx}\right|_{(0,0)} = 0$ for I

at $(0,0)\left.\frac{dy}{dx}\right|_{(0,0)} = \infty$ for II

Hence, angle $= 90°$

now, $\left.\frac{dy}{dx}\right|_{(8,16)} = \frac{3x^2}{4y} = \frac{3}{4}\frac{64}{16} = 3$ for I

$\left.\frac{dy}{dx}\right|_{(8,16)} = \frac{32}{2y} = \frac{16}{16} = 1$ for II

$\therefore \tan\theta = \frac{3-1}{1+3} = \frac{2}{4} = \frac{1}{2}$

$\Rightarrow$ angle between the two curves at the origin is $90°$

c. The two curves are

$xy = a^2$ (1)

$x^2 + y^2 = 2a^2$ (2)

Solving (1) and (2), the points of intersection are $(a,a)$and $(-a,-a)$

Differentiating (1), $dy/dx = -y/x = m_1$ (say)

Differentiating (2), $dy/dx = -x/y = m_2$ (say)

At both points, $m_1 = -1 = m_2$

Hence, the two curves touch each other

d. $y^2 = x, x^3 + y^3 = 3xy$

For the 1st curve, $2y\frac{dy}{dx} = 1 \Rightarrow \left.\frac{dy}{dx}\right|_P = \frac{1}{2y_1}$

Again for the 2nd curve, $\left.\frac{dy}{dx}\right|_P = \frac{y_1 - x_1^2}{y_1^2 - x_1}$

solving $y^2 = x$ and $x^3 + y^3 = 3xy$;

$y^6 + y^3 = 3y^3 \Rightarrow y^3 + 1 = 3 \Rightarrow y^3 = 2$

$\therefore y_1 = 2^{1/3}$ and $x_1 = 2^{2/3}$

Now $m_1 = \frac{1}{2\times 2^{\frac{1}{3}}} = \frac{1}{2^{\frac{4}{3}}}$; $m_2 = \frac{2^{\frac{1}{3}} - 2^{\frac{4}{3}}}{2^{\frac{2}{3}} - 2^{\frac{2}{3}}} \to \infty$

$\tan\theta = \left|\frac{m_2 - m_1}{1+m_1m_2}\right| = \left|\frac{1-\frac{m_1}{m_2}}{\frac{1}{m_2} + m_1}\right| = \left|\frac{1}{m_1}\right| = 2^{\frac{4}{3}}$

$= 16^{\frac{1}{3}}$

$\therefore \theta = \tan^{-1}(16^{\frac{1}{3}})$

286 **(b)**

a. $f(x) = x^2\log x$

For $f'(x) = x(2\log x + 1) = 0, \Rightarrow x = \frac{1}{\sqrt{e}}$ which is the point of minima as derivative changes sign from negative to positive

Also, the function decreases in $\left(0, \frac{1}{\sqrt{e}}\right)$

b. $y = x\log x$

$\Rightarrow \frac{dy}{dx} = x\times\frac{1}{x} + \log x \times 1 = 1 + \log x$ and $\frac{d^2y}{dx^2} = \frac{1}{x}$

For $\frac{dy}{dx} = 0 \Rightarrow \log x = -1 \Rightarrow x = \frac{1}{e}$

$\frac{d^2y}{dx^2} = \frac{1}{1/e} = e > 0$ at $x = \frac{1}{e}$

$\Rightarrow y$ is min for $x = \frac{1}{e}$

c. $f(x) = \frac{\log x}{x}$

For $f'(x) = \frac{1-\log x}{x^2} = 0, x = e$. Also, derivative changes sign from positive to negative at $x = e$, hence it is the point of maxima

d. $f(x) = x^{-x}$

$f'(x) = -x^{-x}(1 + \log x) = 0 \Rightarrow x = 1/e$, which is clearly point of maxima

287 **(c)**

Since $f(x)$ is minimum at $x = -2$ and maximum at $x = 2$, let $\mathrm{g}(x) = ax^3 + bx^2 + cx + d$

$\therefore \mathrm{g}(x)$ is also minimum at $x = -2$ and maximum at $x = 2$

$\therefore a < 0$

$\because a$ is a root of $x^2 - x - 6 = 0$, i.e., $x = 3, -2$

$\therefore a = -2$

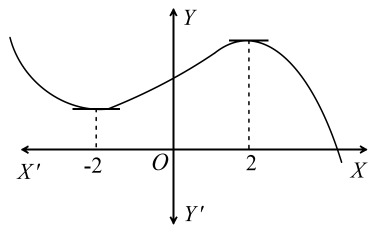
Then, $\mathrm{g}(x) = -2x^3 + bx^2 + cx + d$

$\therefore \mathrm{g}'(x) = -6x^2 + 2bx + c = -6(x+2)(x-2)$

($\because \mathrm{g}(x)$ is minimum at $x = -2$ and maximum at $x = 2$)

On comparing, we get

$b = 0$ and $c = 24$



Since minimum and maximum values are positive

$\therefore \mathrm{g}(-2) > 0 \Rightarrow 16 - 48 + d > 0 \Rightarrow d > 32$

and $\mathrm{g}(2) > 0 \Rightarrow -16 + 48 + d > 0 \Rightarrow d > -32$

It is clear $d > 32$

Hence, $a = -2, b = 0, c = 24, d > 32$

288 **(a)**

a.q,r. $f(x) = (x-1)^3(x+2)^5$

$\Rightarrow f'(x) = 3(x-1)^2(x+2)^5 + 5(x-1)^3(x+2)^4$

$\Rightarrow f'(x) = (x-1)^2(x+2)^4[3(x+2) + 5(x-1)]$

$= (x-1)^2(x+2)^4[8x+1]$

Sign of derivative does not change at $x = 1$ and $x = -2$

Sign of derivative changes sign at $x = -1/8$ from –ve to +ve

Hence, function has point of minima

Also, $f'(x) = 0$ for $x = 1$ and $x = -2$

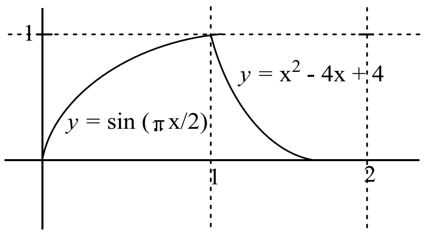
Hence, function has two points of inflection

b. r,s $f(x) = 3\sin x + 4\cos x - 5x$

$\Rightarrow f'(x) = 3\cos x - 4\sin x - 5 \leq 0$, hence $f(x)$ is decreasing function

Also, $f'(x) = -3\sin x - 4\cos x = 0$ for infinite values of x, hence function has infinite points of inflection

c. p,r



From the graph $x = 1$ points of maxima as well as point of inflection

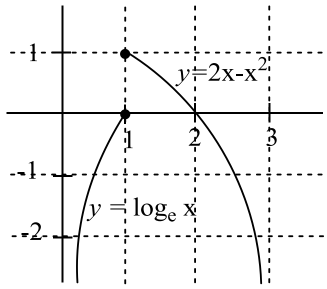
d. r,s $f(x) = (x-1)^{3/5} \Rightarrow f'(x) = \frac{3}{5}(x-1)^{-\frac{2}{5}} \geq 0$ for all real x

Also, $f'(x) = -\frac{3}{5}\frac{2}{5}(x-1)^{-\frac{7}{5}}$ which changes sign at $x = 1$

Hence, $x = 1$ is point of inflection

289 **(b)**

a. r. From the graph $x = 1$ is point of maxima



b. s. $f(x) = \begin{cases} x-1, & x < 2 \\ 0, & x = 2 \\ \sin x, & x > 2 \end{cases}$

$f(2) = 0, f(2^+) = \sin(2^+) > 0$ and $f(2^-) > 0$, hence $x = 2$ is point of minima

c. p. $f(x) = \begin{cases} 2x+3 & x < 0 \\ 5, & x = 0 \\ x^2+7, & x > 0 \end{cases}$

$f(0^-) = 3, f(0) = 5, f(0^+) = 7$, hence

$f(0^-) < f(0) < f(0^+)$

Thus, $f(x)$ is increasing at $x = 0$

d. q. $f(x) = \begin{cases} e^{-x}, & x < 0 \\ 0, & x = 0 \\ -\cos x, & x > 0 \end{cases}$

$f(0) = 0, f(0^+) = -1, f(0^-) = 1$

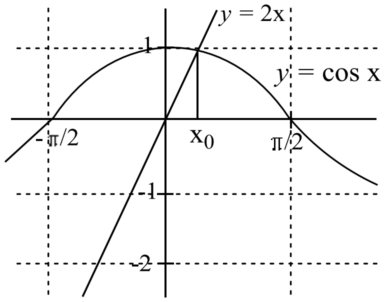
Thus, $f(0^-) > f(0) > f(0^+)$

Hence, $f(x)$ is decreases at $x = 0$

290 **(d)**

$f(x) = \sin x - x^2 + 1$

$f'(x) = \cos x - 2x$



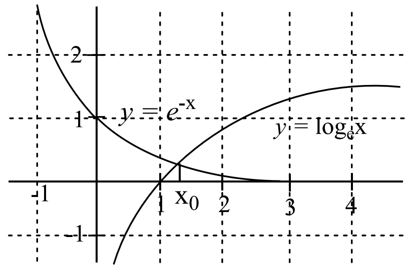
$\Rightarrow f'(x) < 0$ for $x > x_0$

$f'(x) > 0$ for $x < x_0$

Hence $x = x_0$ is point of maxima

b.p $f(x) = x\log_e x - x + e^{-x}$

$f'(x) = \log_e x + 1 - 1 - e^{-x} = \log_e x - e^{-x}$



From the graph for

$x < x_0, e^{-x} > \log_e x, \Rightarrow f'(x) < 0$

For $x > x_0, e^{-x} < \log_e x, \Rightarrow f'(x) > 0$

Hence, $x = x_0$ is point of minima

c.s $f(x) = -x^3 + 2x^2 - 3x + 1$

$f'(x) = -3x^2 + 4x - 3$

Now $D = 16 - 4(-3)(-3) = -20 < 0$

Hence $f'(x) < 0$, for all real x

$\Rightarrow f(x)$ is always decreasing

d.r. $f(x) = \cos \pi x + 10x + 3x^2 + x^3$

$\Rightarrow f'(x) = -\pi \sin \pi x + 10 + 6x + 3x^2$

$= 3(x^2 + 2x + 10/3) - \pi \sin \pi x$

$= 3((x+1)^2 + 7/3) - \pi \sin \pi x$

Now min. value of $3((x+1)^2 + 7/3)$ is 7 but maximum value of $\pi \sin \pi x$ is π

Hence, $f'(x) > 0$ for all real x

Hence, $f(x)$ is always increasing

291 **(b)**

$\because \quad f''(-1) = f''\left(\frac{1}{3}\right) = 0$

$\therefore \quad f''(x) = a(x+1)\left(x - \frac{1}{3}\right)$

$= a''\left(x^2 + \frac{2}{3}x - \frac{1}{3}\right)$

On integrating w.r.t. x, we get

$f(x) = \frac{a''}{3}(x^3 + x^2 - x) + \lambda \quad$ ...(i)

Where λ is constant of integration

And $f(-2) = 0$

$$\Rightarrow \frac{a''}{3}(-8+4+2)+\lambda = 0$$

$$\therefore \lambda = \frac{2a''}{3}$$

From Eq. (i),

$$f(x) = \frac{a''}{3}(x^3+x^2-x+2)$$

Also, $\int_{-1}^{1} f(x)\,dx = \frac{14}{3}$

$$\Rightarrow \frac{a''}{3}\int_{-1}^{1}(x^3+x^2-x+2)\,dx = \frac{14}{3}$$

$$\Rightarrow 0+\frac{2a''}{3}\int_{0}^{1}(x^2-2)\,dx = \frac{14}{3}$$

$$\Rightarrow \frac{2a''}{3}\left\{\frac{1}{3}+2\right\} = \frac{14}{3}$$

$$\Rightarrow \frac{14a''}{9} = \frac{14}{3}$$

$$\Rightarrow a'' = 3$$

Then, $f(x) = x^3 - x^2 - x + 2$

On comparing with $f(x) = ax^3 + bx^2 + cx + d$

$\therefore a = 1, b = 1, c = -1, d = 2$

292 **(b)**

Let y be the length and x be the breadth of the rectangular portion. Total perimeter of the window is

$2x + 2y + \left(\frac{1}{2}\right)\pi y = P$ (say)

Let amount of light per square meter for the coloured glass beμ. If L is the total light transmitted, then

$L = 3\mu \times$ Area of rectangle portion+ $\mu \times$ Area of semi-circular portion

$$= 3\mu xy + \frac{1}{8}\mu\pi y^2$$

$$= \mu\left[\frac{3}{2}y\left(P-\left(2+\frac{\pi}{2}\right)y\right)+\frac{1}{8}\pi y^2\right]$$

$$\Rightarrow \frac{dL}{dy} = \frac{\mu}{2}\left[3P-6\left(2+\frac{\pi}{2}\right)y+\frac{\pi}{2}y\right]$$

Put $\frac{dL}{dy} = 0$

$$\Rightarrow y = \frac{3P}{12+\frac{5\pi}{2}}$$

$$\frac{d^2L}{dy^2} = \frac{\mu}{2}\left(-12-\frac{5\pi}{2}\right) < 0$$

$$\therefore \frac{y}{x} = \frac{2.\frac{3P}{12+5\pi/2}}{P-\left(2+\frac{\pi}{2}\right).\frac{3P}{12+\frac{5\pi}{2}}}$$

$$= \frac{6}{(12+5\pi/2)-3(2+\pi/2)} = \frac{6}{6+\pi}$$

293 **(a)**

$$f(x) = \frac{(x^2+ax+1)-2ax}{x^2+ax+1}$$

$$= 1-\frac{2ax}{x^2+ax+1}$$

$$\therefore f''(x) = -\left[\frac{(x^2+ax+1).2a-2ax(2x+a)}{(x^2+ax+1)^2}\right]$$

$$= -\left[\frac{-2ax^2+2a}{(x^2+ax+1)^2}\right]$$

$$= 2a\left[\frac{(x^2+1)}{(x^2+ax+1)^2}\right] \quad ...(i)$$

and

$f''''(x) =$

$$2a\left[\frac{(x^2+ax+1)^2(2x)-2(x^2-1)(x^2+ax+1)(2x+a)}{(x^2+ax+1)^4}\right]$$

$$= 2a\left[\frac{2x(x^2+ax+1)-2(x^2-1)(2x+a)}{(x^2+ax+1)^3}\right]$$

Now, $f''''(1) = \frac{4a(a+2)}{(a+2)^3} = \frac{4a}{(a+2)^2}$

And $f(-1) = \frac{4a(a-2)}{(2-a)^3} = \frac{4a}{(a-22)^2}$

$\therefore (2+a)^2 f''''(1) + (2-a)^2 f''''(-1) = 4a - 4a$
$= 0$

294 **(a)**

Let $P_1(t_1, t_1^3)$ is a point on the curve $y = x^3$

$$\therefore \left.\frac{dy}{dx}\right|_{(t_1,t_1^3)} = 3t_1^2$$

Tangent at P_1 is $y - t_1^3 = 3t_1^2(x - t_1)$ (1)

The intersection of (1) and $y = x^3$

$\Rightarrow x^3 - t_1^3 = 3t_1^2(x - t_1)$

$\Rightarrow (x-t_1)(x^2 + xt_1 + t_1^2) - 3t_1^2(x-t_1) = 0$

$\Rightarrow (x-t_1)^2(x+2t_1) = 0$

If $P_2(t_2, t_2^3)$, then

$(t_2 - t_1)^2(t_2 + 2t_1) = 0$

$\therefore t_2 = -2t_1 \ (t_2 \neq t_1)$

Similarly, the tangent at P_2 will meet the curve at the point

$P_3(t_3\, t_3^3)$ when $t_3 = -2t_2 = 4t_1$ and so on

The abscissa of $P_1, P_2, \ldots, P_n$ are

$t_1, -2t_1, 4t_1, \ldots, (-2)^{n-1}t_1$ in G.P

$\therefore \frac{t_2}{t_1} = \frac{t_3}{t_2} = \frac{t_4}{t_3} = \cdots = -2$ (r say)

$\therefore t_2 = t_1 r, t_3 = t_2 r$ and $t_4 = t_3 r$

If $x_1 = 1$, then $x_2 = -2, t_3 = 4, \ldots$

Then $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{x_n}$ = sum of infinite G.P. with common ratio $(-1/2)$ with first term 1

$= \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{2}{3}$

295 **(a)**

$\frac{dy}{dx} = \frac{1-9t^2}{-6t} = \tan\theta$

$\Rightarrow 9t^2 - 6\tan\theta, t - 1 = 0$

$\Rightarrow 3t = \tan\theta \pm \sec\theta$

$\Rightarrow \tan\theta + \sec\theta = 3t$

296 **(b)**

Let V be the volume and r the radius of the balloon at any time, then

$V = \left(\frac{4}{3}\right)\pi r^3$

$\Rightarrow \frac{dV}{dt} = \left(\frac{4}{3}\right)(3\pi r^2)\frac{dr}{dt}$

$\Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} = 40$ (given)

$\Rightarrow \frac{dr}{dt} = \frac{10}{\pi r^2}$ (1)

Now let S be the surface area of the balloon when its radius is r, then $S = 4\pi r^2$

$\Rightarrow \frac{dS}{dt} = 8\pi r \frac{dr}{dt}$ (2)

From (1) and (2), $\frac{dS}{dt} = 8\pi r \frac{10}{\pi r^2} = \frac{80}{r}$

When $r = 8$, the rate of increase of $S = \frac{80}{8} =$ $10\ \text{cm}^2/\text{min}$

$\Rightarrow$ Increase of S in $\frac{1}{2}$ minute $= 10 \times \left(\frac{1}{2}\right) =$ $5\ \text{cm}^2/\text{min}$

If r_1 is the radius of the balloon after $(1/2)$ min, then $4\pi r_1^2 = 4\pi(8)^2 + 5$

Or $r_1^2 - 8^2 = \frac{5}{4\pi} = 0.397$ nearly or $r_1^2 = 64.397$ or $r_1 = 8.025$ nearly

$\Rightarrow$ Required increase in the radius $= r_1 - 8 = 8.025 - 8$

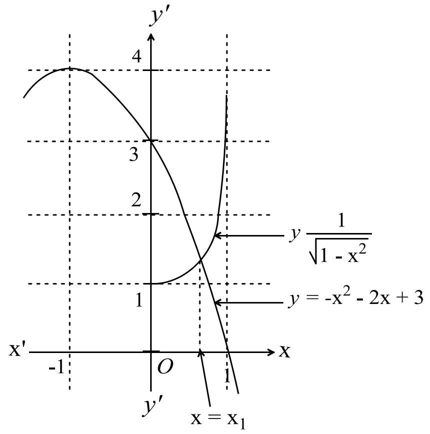
$= 0.025$ cm

297 **(b)**

Let $f(x) = \sin^{-1}x + x^2 - 3x + \frac{x^3}{3}$

$\Rightarrow f'(x) = \frac{1}{\sqrt{1-x^2}} + 2x - 3 + x^2$

$\Rightarrow f'(x) = 0$ for some $x = x_1 \in (0,1)$



and $f''(x) = \frac{x}{(1-x^2)^{3/2}} + 2 + 2x > 0, \forall x \in (0,1)$

$\Rightarrow x = x_1$ is the point of minimum

$f(x)$ is continuous $\forall x \in [0,1]$

Hence, the global maxima exist at $x = 0$ or $x = 1$

$f(0) = 0, f(1) = \pi/2 - 5/3 < 0$

$f(0)$ is global maxima $\forall x \in [0,1]$

$\Rightarrow f(x) \le f(0), x \in [0,1]$

$\Rightarrow \sin^{-1}x + x^2 - 3x + x^3/3 \le 0$

$\Rightarrow \sin^{-1}x + x^2 \le \frac{x(9-x^2)}{3}\ \forall x \in [0,1]$

298 **(a)**

$g'(x) = f'(\sin x)\cos x - f'(\cos x)\sin x$

$\Rightarrow g''(x) = -f'(\sin x)\sin x + \cos^2 x\, f''(\sin x)$
$+ f''(\cos x)\sin^2 x$
$- f'(\cos x) > 0\ \ \forall x \in (0, \pi/2)$

(as it is given $f'(\sin x) = f'(\cos x\,(\pi/2 - x)) < 0$ and $f''(\sin x)$

$= f''(\cos x\,(\pi/2 - x)) > 0$)

$\Rightarrow g'(x)$ is increasing in $(0, \pi/2)$. Also $g'(\pi/4) = 0$

$\Rightarrow g'(x) > 0\ \forall x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ and $g'(x) < 0\ \forall x \in (0, \pi/4)$

Thus $g(x)$ is decreasing in $(0, \pi/4)$

299 **(d)**

If $f(x)$ is continuous then $f(3^-) = f(3^+) \Rightarrow -9 + 12 + a$

$= 3a + b \Rightarrow 2a + b = 3$ (1)

Also $f(4^-) = f(4^+) \Rightarrow 4a + b = -b + 6 \Rightarrow 2a + b = 3$ (2)

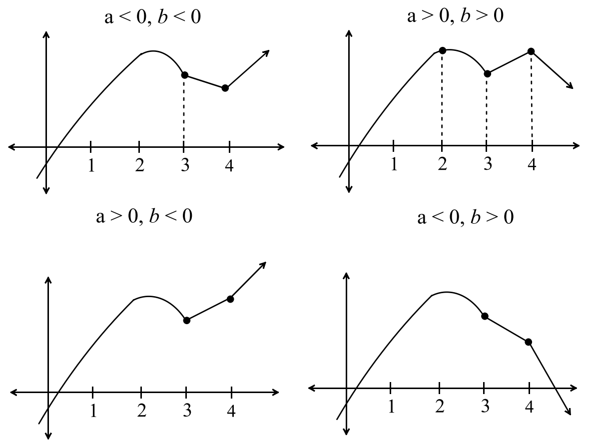
$\Rightarrow f(x)$ is continuous for infinite values of a and b

Also, $f'(x) = \begin{cases} -2x + 4, & x < 3 \\ a, & 3 < x < 4 \\ -\frac{b}{4}, & x > 4 \end{cases}$. For $f(x)$ to be differentiable, $f'(3^-) = f(3^+) \Rightarrow a = -2$ and

$-\frac{b}{4} = a = -2$

$\Rightarrow b = 8$

Hence, $f(x)$ can be differentiable



300 **(a)**

$\frac{dP(x)}{dx} > P(x)$

$\Rightarrow e^{-x}\frac{dP(x)}{dx} - e^{-x}P(x) > 0$

$\Rightarrow \frac{d}{dx}(P(x)e^{-x}) > 0$

$\Rightarrow P(x)e^{-x}$ is an increasing function

$\Rightarrow P(x)e^{-x} > P(1)e^{-1} \forall x \geq 1$

$\Rightarrow P(x)e^{-x} > 0 \ \forall x > 1 \Rightarrow P(x) > 0 \ \ \forall x > 1$

301 **(a)**

$h(x) = f(x) - a(f(x))^2 + a(f(x))^3$

$\Rightarrow h'(x) = f'(x) - 2af(x)f'(x) + 3a(f(x))^2 f'(x)$

$\Rightarrow h'(x) = f'(x)\left[3a(f(x))^2 - 2af(x) + 1\right]$

Now $h(x)$ increases if $f(x)$ increases and $3a(f(x))^2 - 2af(x) + 1 > 0$ for all $x \in R$

$\Rightarrow 3a > 0$ and $4a^2 - 12a \leq 0$

$\Rightarrow a > 0$ and $a \in [0, 3]$

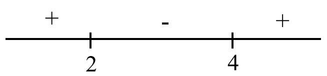
$\Rightarrow a \in [0, 3]$

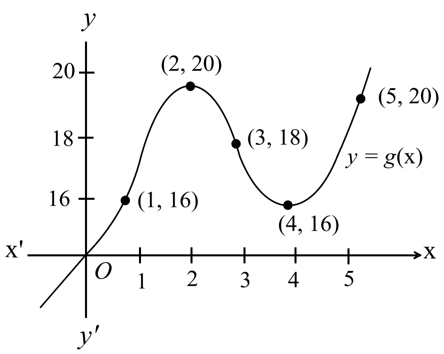
302 **(a)**

Let $g(x) = x^3 - 9x^2 + 24x = x(x^2 - 9x + 24)$

$\Rightarrow g'(x) = 3(x-2)(x-4)$

Sign scheme of $g(x)$





For three real roots of $f(x) = x^3 - 9x^2 + 24x + c = 0$, c must lie in the interval $(-20, -16)$

$f(0) = c < 0$

$f(1) = 1 - 9 + 24 + c = c + 16 < 0$ for $\forall c \in (-20, -16)$

$f(2) = 8 - 36 + 48 + c = c + 20 > 0$

$\alpha \in (1, 2) \Rightarrow [\alpha] = 1$

$f(3) = 27 - 81 + 72 + c = 18 + c$

$\Rightarrow f(3) < 0$ if $c \in (-20, -18)$ or $f(3) > 0$ if $c \in (-18, -16)$

or $\beta \in (2, 3)$ if $c \in (-20, -18)$

and $\beta \in (3, 4)$ if $c \in (-18, -16)$

Now $f(4) = 64 - 144 + 96 + c = 16 + c < 0 \ \forall c \in (-20, -16)$

$f(5) = 125 - 225 + 120 + c = c + 20 > 0 \ \forall c \in (-20, -16)$

$\Rightarrow \gamma \in (4, 5) \Rightarrow [\gamma] = 4$

Thus,

$[\alpha] + [\beta] + [\gamma] = \begin{cases} 1 + 2 + 4, -20 < c < -18 \\ 1 + 3 + 4, -18 < c < -16 \end{cases}$

Now if $c \in (-20, -18)$

$\alpha \in (1, 2), \beta \in (2, 3), \gamma \in (4, 5)$

$\Rightarrow [\alpha] + [\beta] + [\gamma] = 7$

If $c \in (-18, -16), \alpha \in (1, 2), \beta \in (3, 4), \gamma \in (4, 5)$

$\Rightarrow [\alpha] + [\beta] + [\gamma] = 8$

303 **(b)**

$f'(x) \leq 0 \ \forall x \in [a, b]$, so $f(x)$ is a decreasing function and $f(c) = 0 \Rightarrow f(x)$ cuts x-axis once when $x = c$

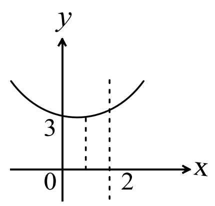
304 **(b)**

$f(x) = 4x^2 - 4ax + a^2 - 2a + 2$. Vertex of this parabola is $\left(\frac{a}{2}, 2 - 2a\right)$

Case 1: $0 < \frac{a}{2} < 2$

In this case, $f(x)$ will attain the minimum value at $x = \frac{a}{2}$

Thus, $f\left(\frac{a}{2}\right) = 3$



$\Rightarrow 3 = -2a + 2 \Rightarrow a = -\frac{1}{2}$ (Rejected)

Case 2: $\frac{a}{2} \geq 2$

In this, $f(x)$ attains the global minimum value at $x = 2$

Thus $f(2) = 3$

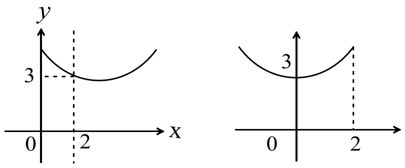
$\Rightarrow 3 = 16 - 8a + a^2 - 2a + 2 \Rightarrow a = 5 \pm \sqrt{10}$

Thus $a = 5 + \sqrt{10}$

Case 3: $\frac{a}{2} \le 0$

In this case, $f(x)$ attains the global minimum value at $x = 0$. Thus $f(0) = 3$

Convert the following graph



$\Rightarrow 3 = a^2 - 2a + 2 \Rightarrow a = 1 \pm \sqrt{2}$. Thus, $a = 1 - \sqrt{2}$

Hence, the permissible values of a are $1 - \sqrt{2}$ and $5 + \sqrt{10}$

$f(x) = 4x^2 - 49x + a^2 - 2a + 2$ is monotonic in $[0, 2]$

Hence, the point of minima of function should not lie in $[0, 2]$

Now $f'(x) = 0 \Rightarrow 8x - 4a = 0 \Rightarrow x = a/2$. If $\frac{a}{2} \in [0, 2]$

$\Rightarrow a \in [0, 4]$

For $f(x)$ to be monotonic in $[0, 2], a \notin [0, 4] \Rightarrow$ $a \le 0$ or $a \ge 4$

305 **(a)**

$f(x) = x^3 - 3(7 - a)x^2 - 3(9 - a^2)x + 2$

$\Rightarrow f'(x) = 3x^2 - 6(7 - a)x - 3(9 - a^2)$

For real root $D \ge 0$,

$\Rightarrow 49 + a^2 - 14a + 9 - a^2 \ge 0 \Rightarrow a \le \frac{58}{14}$ (1)

When point of minima is negative, point of maxima is also negative

Hence, equation $f'(x) = 3x^2 - 6(7 - a)x - 3(9 - a^2) = 0$ has both roots negative

For which sum of roots $= 2(7 - a) < 0$ or $a > 0$, which is not possible as from (1), $a \le \frac{58}{14}$

When point of maxima is positive, point of minima is also positive

Hence, equation $f'(x) = 3x^2 - 6(7 - a)x - 3(9 - a^2) = 0$ has both roots positive

For which sum of roots $= 2(7 - a) > 0 \Rightarrow a < 7$ (2)

Also product of roots is positive $\Rightarrow -(9 - a^2) > 0$ or $a^2 > 9$

or $a \in (-\infty, -3) \cup (3, \infty)$ (3)

From (1), (2) and (3); $a \in (-\infty, -3) \cup (3, 58/14)$

For points of extrema of opposite sign, equation (1) has roots of opposite sign

$\Rightarrow a \in (-3, 3)$

306 **(c)**

$f(x) = \left(1 + \frac{1}{x}\right)^x$, $f(x)$ is defined if $1 + \frac{1}{x} > 0$

$\Rightarrow \frac{x + 1}{x} > 0 \Rightarrow (-\infty, -1) \cup (0, \infty)$

Now $f'(x) = \left(1 + \frac{1}{x}\right)^x \left[\text{In}\left(1 + \frac{1}{x}\right) + \frac{x}{1 + \frac{1}{x}} \frac{-1}{x^2}\right]$

$= \left(1 + \frac{1}{x}\right)^x \left[\text{In}\left(1 + \frac{1}{x}\right) - \frac{1}{x + 1}\right]$

Now $\left(1 + \frac{1}{x}\right)^x$ is always positive, hence the sign of $f'(x)$ depends on sign of $\text{In}\left(1 + \frac{1}{x}\right) - \frac{1}{1 + x}$

Let $g(x) = \text{In}\left(1 + \frac{1}{x}\right) - \frac{1}{x + 1}$

$g'(x) = \frac{1}{1 + \frac{1}{x}} \frac{-1}{x^2} + \frac{1}{(x + 1)^2} = \frac{-1}{x(x + 1)^2}$

(1) for $x \in (0, \infty), g'(x) < 0$

$\Rightarrow g(x)$ is monotonically decreasing for $x \in (0, \infty)$

$\Rightarrow g(x) > \lim_{x \to \infty} g(x)$

$\Rightarrow g(x) > 0$

and since $g(x) > 0 \Rightarrow f'(x) > 0$

(2) for $x \in (-\infty, -1), g'(x) > 0$

$\Rightarrow g(x)$ is monotonically increasing for $x \in (-\infty, -1)$

$\Rightarrow g(x) > \lim_{x \to \infty} g(x)$

$\Rightarrow g(x) > 0 \Rightarrow f'(x) > 0$

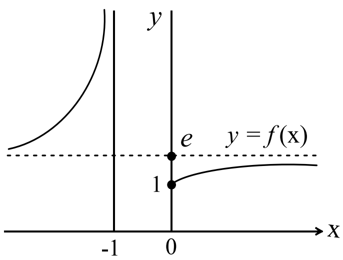
Hence from (1) and (2) we get $f'(x) > 0$ for all $x \in (-\infty, -1) \cup (0, \infty)$

$\Rightarrow f(x)$ is monotonically increasing in its domain

Also $\lim_{x \to \pm\infty} \left(1 + \frac{1}{x}\right)^x = e$

$\lim_{x \to 0^+} \left(1 + \frac{1}{x}\right)^x = 1$ and $\lim_{x \to -1} \left(1 + \frac{1}{x}\right)^x = \infty$

The graph of $f(x)$ is shown in figure



Range is $y \in (1, \infty) - \{e\}$

307 **(d)**

$f(x) = x + \cos x - a \Rightarrow f'(x) = 1 - \sin x \geq 0\ \forall x \in R$

Thus $f(x)$ is increasing in $(-\infty, \infty)$, as for $f'(x) = 0, x$ is not forming an interval

Also $f''(x) = -\cos x = 0$

$\Rightarrow x = (2n+1)\dfrac{\pi}{2}, n \in Z$

Hence infinite points of inflection

Now $f(0) = 1 - a$

For positive root $1 - a < 0 \Rightarrow a > 1$. For negative root $1 - a > 0 \Rightarrow a < 1$

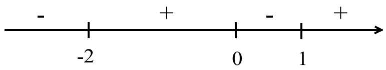
308 **(c)**

$f(x) = 3x^4 + 4x^3 - 12x^2$

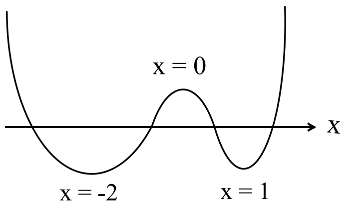
$\Rightarrow f'(x) = 12\,(x^3 + x^2 - 2x)$

$= 12x(x-1)(x+2)$

The sign scheme of $f'(x)$ is as follows



The graph of the function is as follows



Thus, we have,

$f(-2) = -32$ and $f(1) = -5$

Hence, range of the function is $[-32, \infty)$

Also, $f(x) =$ a has no real roots if $a < -32$

309 **(d)**

$$f(x) = \frac{x^2 - 6x + 4}{x^2 + 2x + 4}$$

$$= 1 - \frac{8x}{x^2 + 2x + 4}$$

$$f'(x) = -8\left[\frac{(x^2 + 2x + 4) - x(2x+2)}{(x^2 + 2x + 4)^2}\right]$$

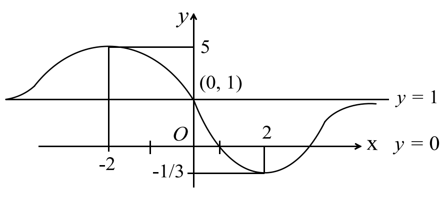
$$= -8\left[\frac{-x^2 + 4}{(x^2 + 2x + 4)^2}\right] = \frac{8(x^2 - 4)}{(x^2 + 2x + 4)^2}$$

$f'(x) = 0 \Rightarrow x = 2$ or -2

$$f(2) = \frac{4 - 12 + 4}{4 + 4 + 4} = \frac{-4}{12} = \frac{-1}{3}$$

$$f(-2) = \frac{4 + 12 + 4}{4 - 4 + 4} = 5$$

The graph of $y = f(x)$ is as shown



Hence $-\frac{1}{3} \leq f(x) \leq 5$

310 **(c)**

Since two points of inflection occur at $x = 1$ and $x = 0$

$\Rightarrow P''(1) = P''(0) = 0$

$\therefore P''(x) = a(x^2 - x)$

$\Rightarrow P'(x) = a\left(\dfrac{x^3}{3} - \dfrac{x^2}{2}\right) + b$

Also, Given $\left(\frac{dy}{dx}\right)_{x=0} = \sec^{-1}\sqrt{2} = \tan^{-1} 1$

Hence, $P'(0) = 1$, so $b = 1$

$\Rightarrow P'(x) = a\left(\dfrac{x^3}{3} - \dfrac{x^2}{2}\right) + 1$

$\therefore P(x) = a\left(\dfrac{x^4}{12} - \dfrac{x^3}{6}\right) + x + c$

As $P(-1) = 1$

$\Rightarrow a\left(\frac{1}{12} + \frac{1}{6}\right) - 1 + c = 1 \Rightarrow \frac{a}{4} + c = 2 \quad (1)$

$P(1) = 2$

$\Rightarrow a\left(\dfrac{1}{12} - \dfrac{1}{6}\right) + 1 + c = 1$

$\Rightarrow -\dfrac{a}{12} + c = 0 \quad (2)$

Solving (1) and (2),

We have $a = 6$ and $c = \frac{1}{2}$

$\Rightarrow P(x) = 6\left(\dfrac{x^4}{12} - \dfrac{x^3}{6}\right) + x + \dfrac{1}{2}$

$\Rightarrow P(-1) = 6\left(\frac{1}{12} + \frac{1}{6}\right) - 1 + \frac{1}{2} = 1$ and $P(0) = \frac{1}{2}$

$\Rightarrow P'(x) = 6\left(\dfrac{x^3}{3} - \dfrac{x^2}{2}\right) + 1 = (x-1)^2(2x+1)$

311 **(c)**

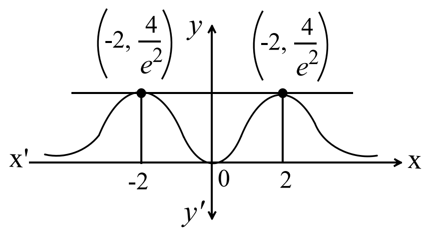
We have $f(x) = x^2 e^{-|x|} = \begin{cases} x^2 e^{-x}, & x \geq 0 \\ x^2 e^{x}, & x < 0 \end{cases}$

$\Rightarrow f'(x) = \begin{cases} e^{-x}(2x - x^2), x \geq 0 \\ e^{x}(x^2 + 2x), x < 0 \end{cases}$

$f(x)$ increases in $(-\infty, -2) \cup (0, 2)$

and $f(x)$ decreases in $(-2, 0) \cup (2, \infty)$

$\Rightarrow f'(x) = \begin{cases} e^{-x}(x^2 - 4x + 2), x \geq 0 \\ e^{x}(x^2 + 4x + 2), x < 0 \end{cases}$



$f'(x) = 0$ has four roots. Hence, four points of inflection

312 **(2)**

$y = x^n$

$\dfrac{dy}{dx} = n\,x^{n-1} = na^{n-1}$

Slope of normal $= -\frac{1}{na^{n-1}}$

Equation of normal $y - a^n = -\frac{1}{na^{n-1}}(x-a)$

Put $x = 0$ to get y-intercept

$y = a^n + \frac{1}{na^{n-2}}$; Hence $b = a^n + \frac{1}{na^{n-2}}$

$$\lim_{a\to 0} b = \begin{cases} 0 & \text{if} \quad n < 2 \\ \frac{1}{2} & \text{if} \quad n = 2 \\ \infty & \text{if} \quad n > 2 \end{cases}$$

313 **(9)**

$$f\left(\frac{3}{2}\right) = 0 \Rightarrow \lim_{x\to\frac{3}{2}} |x^2 - 3x| + a \le 0 \Rightarrow a \le -\frac{9}{4}$$

Hence, greatest value of $|4a|$ is 9

314 **(6)**

$f(x) = f(6-x) \quad (1)$

On differentiating (1) w.r.t. x, we get

$f'(x) = -f'(6-x) \qquad (2)$

Putting $x = 0, 2, 3, 5$ in (2), we get

$f'(0) = -f'(6) = 0$

Similarly $f'(2) = -f'(4) = 0$

$f'(3) = 0$

$f'(5) = -f'(1) = 0$

$\therefore f'(0) = 0 = f'(2) = f'(3) = f'(5) = f'(1)$
$= f'(4) = f'(6)$

$\therefore$ $f'(x) = 0$ has minimum 7 roots in [0, 6]

Now, consider a function $y = f'(x)$

As $f'(x)$ satisfy Rolle's theorem in intervals $[0,1], [1,2], [2,3], [3,4], [4,5]$ and $[5,6]$ respectively

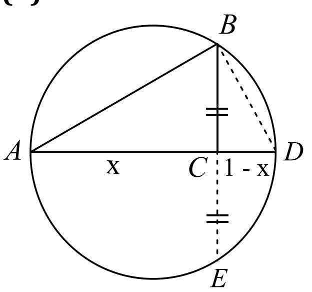
So, by Rolle's theorem, the equation $f''(x) = 0$ has minimum 6 roots

Now $g(x) = (f''(x))^2 + f'(x)f'''(x) == h'(x)$, where $h(x) = f'(x)f''(x)$

Clearly $h(x) = 0$ has minimum 13 roots in [0, 6]

Hence again by Rolle's theorem, $g(x) = h'(x)$ has minimum 12 zeroes in [0, 6]

315 **(9)**



$BC \times CE = AC \times CD$

$\Rightarrow (BC)(CE) = x(1-x)$

But $BC = CE$

$\therefore BC = \sqrt{x(1-x)}$

$\Rightarrow$ Area $\Delta = \dfrac{x\sqrt{x - x^2}}{2}$

$\Rightarrow \Delta^2 = \dfrac{x^3 - x^4}{2}$

$\Rightarrow \dfrac{d\Delta^2}{dx} = \dfrac{3x^2 - 4x^3}{2}$

If $\frac{d\Delta^2}{dx} = 0$

$\Rightarrow x = 3/4$ which is the point of maxima

Hence, maximum area is $\frac{3\sqrt{3}}{32}$

316 **(2)**

Given $\lim_{x\to 0}\left(\frac{P(x)}{x^3} - 2\right) = 4$

$\therefore \lim_{x\to 0} \dfrac{P(x)}{x^3} = 6$

Consider $P(x) = ax^5 + bx^4 + 6x^3$

$\Rightarrow P'(x) = 5ax^4 + 4bx^3 + 18x^2$

Now, $P'(-1) = 0 \Rightarrow 5a - 4b = -18$

and $P'(1) = 0 \Rightarrow 5a + 4b = -18$

$\therefore$ On solving, we get $a = \frac{-18}{5}, b = 0$

Hence, $P(x) = \frac{-18}{5}x^5 + 6x^3$

$\Rightarrow P(1) = \dfrac{12}{5}$

317 **(4)**

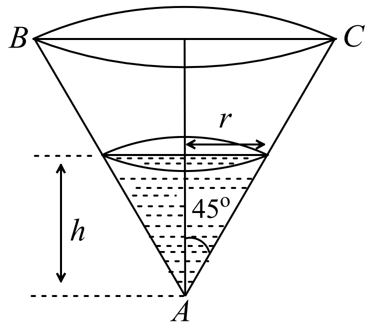
$f''(x) = 12x^2 + 6ax + 3 \ge 0 \; \forall x \in R$

$\Rightarrow 36a^2 - 144 \le 0$

$\Rightarrow a \in [-2, 2]$

$\Rightarrow$ Number of non-zero integral values of 'a' is 4

318 **(5)**



We have

$\dfrac{dV}{dt} = 2 \Rightarrow \dfrac{d}{dt}\left(\dfrac{1}{3}\pi r^3\right)$

$= 2$ [Here $r = h$, as $\theta = 45°$]

$\Rightarrow \pi r^2 \frac{dr}{dt} = 2 \Rightarrow \frac{dr}{dt} = \frac{2}{\pi r^2} \qquad (1)$

Now, perimeter $= 2\pi r = p$(let)

$\Rightarrow \frac{d}{dt}(2\pi r) = 2\pi\left(\frac{2}{\pi r^2}\right) = \frac{4}{r^2}$ (2) (Using equation (1))

When $h = 2$ m $\Rightarrow r = 2$ m

Hence $\frac{dp}{dt} = \frac{4}{4} = 1$ m/s

319 **(3)**

$f''(x) = 4x$

$f'(x) = 2x^2 + C$

Given $f'(-2) = 1 \Rightarrow C = -7$

$\therefore f'(x) = 2x^2 - 7$

$f(x) = \frac{2}{3}x^3 - 7x + C, f(-2) = 0$

$0 = -\frac{16}{3} + 14 + C \Rightarrow C = -\frac{26}{3}$

$\therefore f(x) = \frac{2}{3}x^3 - 7x - \frac{26}{3} = \frac{1}{3}(2x^3 - 21x - 26)$

$\therefore f(1) = -15$

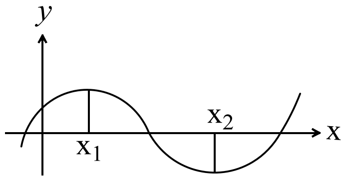
320 **(1)**

$f(x) = \frac{x^3}{3} - x - b;$

$\therefore f'(x) = x^2 - 1 = 0$

$\therefore x = 1$ or -1

For three distinct roots $f(x_1) \cdot f(x_2) < 0$ where x_1 and x_2 are the roots of $f'(x) = 0$

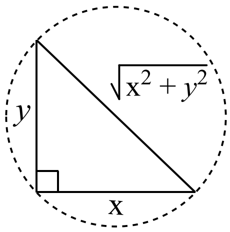


$\Rightarrow \left(\frac{1}{3} - 1 - b\right)\left(-\frac{1}{3} + 1 - b\right) < 0$

$\Rightarrow \left(b + \frac{2}{3}\right)\left(b - \frac{2}{3}\right) < 0$

$\Rightarrow b \in \left(-\frac{2}{3}, \frac{2}{3}\right)$

321 **(9)**



$\frac{9}{\pi} = S = \frac{xy}{2} =$ constant

Area of the circles $A(x) = \pi r^2 = \frac{\pi(x^2+y^2)}{4}; (x^2 + y^2 = 4r^2)$

$A(x) = \frac{\pi}{4}\left[x^2 + \left(\frac{2S}{x}\right)^2\right]$

$A'(x) = \frac{\pi x}{2} - \frac{2\pi S^2}{x^3} = 0$

$\Rightarrow x^4 = 4S^2$

$\Rightarrow x^2 = 2S$

$\Rightarrow S^2 = \frac{x^2y^2}{4} = \frac{2Sy^2}{4}$

$\Rightarrow y^2 = 2S$

Therefore, least area of circle $= \pi r^2 = \frac{\pi}{4}(x^2 + y^2) = \pi S = 9$ sq. units

322 **(9)**

Let $y = 2x \tan^{-1} x - \text{In}(1 + x^2)$

$y' = 2\tan^{-1} x + \frac{2x}{1 + x^2} - \frac{2x}{1 + x^2}$

$\Rightarrow y' > 0 \ \forall\, x \in R^+, y' < 0 \ \forall x \in R^-$

$\Rightarrow y \geq 0, \forall x \in R$

$\therefore 4 - |[x]|$ takes the values 0, 1, 2, 3, 4

$\{\because |\alpha| \leq 4 - |[x]|\}$

$|\alpha| \leq 4 = |[x]|$ is satisfied by

$\alpha = 0, \pm 1, \pm 2, \pm 3, \pm 4,$

Therefore, number of values of α is 9

323 **(3)**

We have $f(x, y) = x^2 + y^2 - 4x + 6y$

Let $(x, y) = (\cos\theta, \sin\theta)$, then $\theta \in [0, \pi/2]$ and

$f(x, y) = f(\theta) = \cos^2\theta + \sin^2\theta - 4\cos\theta + 6\sin\theta$

$f'(\theta) = 6\cos\theta + 4\sin\theta > 0 \ \forall\theta \in [0, \pi/2]$

$\therefore f'(\theta)$ is strictly increasing in $[0, \pi/2]$

$\therefore f(\theta)_{\min} = f(0) = 1 - 4 + 0 = -3$

324 **(4)**

We have $f(0) = 2$

Now $y - f(a) = f'(a)[x - a]$

For x intercept $y = 0$, so

$x = a - \frac{f(a)}{f'(a)} = a - 2 \Rightarrow \frac{f(a)}{f'(a)} = 2$

$\Rightarrow \frac{f'(a)}{f(a)} = \frac{1}{2}$

$\therefore$ On integration both sides w.r.t. a, we get

In $f(a) = \frac{a}{2} + C$

$f(a) = Ce^{a/2}$

$f(x) = Ce^{x/2}$

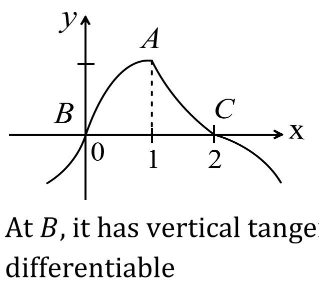
$f(0) = C \Rightarrow C = 2$

$\therefore f(x) = 2e^{x/2}$

Hence $k = 2, p = \frac{1}{2} \Rightarrow \frac{k}{p} = 4$

325 **(3)**

A, B, C are the 3 critical points of $y = f(x)$



At B, it has vertical tangent, hence non-differentiable

At A, it is non-differentiable

At $C, \frac{dy}{dx} = 0$

326 **(9)**

$y = ax^2 + bx + c; \frac{dy}{dx} = 2ax + b$

When $x = 1, y = 0 \Rightarrow a + b + c = 0$ (1)

$\frac{dy}{dx}\Big|_{x=1} = 3$ and $\frac{dy}{dx}\Big|_{x=3} = 1$

$2a + b = 3$ (2)

$6a + b = 1$ (3)

Solving (1), (2) and (3)

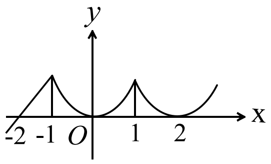
$a = -\dfrac{1}{2}; b = 4, c = -\dfrac{7}{2}$

$\therefore 2a - b - 4c = -1 - 4 + 14 = 9$

327 **(4)**

$f'(x) = \begin{cases} 1, x < -1 \\ 2x, -1 < x < 1 \\ 2(x-2), x > 1 \end{cases}$ $f'(x)$ changes sign at

$x = -1, 0, 1, 2$



328 **(3)**

$\frac{dy}{dx} = \frac{y}{x} = -\frac{1}{2}\cot^3\theta = -\frac{1}{2}$ at $\theta = \frac{\pi}{4}$

Also the point P for $\theta = \pi/4$ is (2, 1)

Equation of tangent is $y - 1 = -\frac{1}{2}(x-2)$

or $x + 2y - 4 = 0$ (1)

This meets the curve whose Cartesian equation on eliminating θ by $\sec^2\theta - \tan^2\theta = 1$ is

$y^2 = \frac{1}{x-1}$ (2)

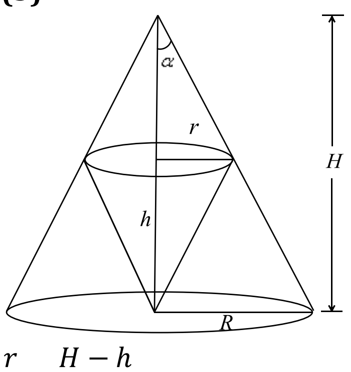
Solving (1) and (2), we get $y = 1, -\frac{1}{2}$

$\therefore x = 2, 5$

Hence P is (2,1) as given and Q is $\left(5, -\frac{1}{2}\right)$

$\therefore PQ = \sqrt{\dfrac{45}{4}} = \dfrac{3\sqrt{5}}{2}$

329 **(3)**



$\dfrac{r}{R} = \dfrac{H-h}{H}$

$r = \dfrac{R(H-h)}{H}$

Volume $V = \frac{1}{3}\pi\frac{R^2(H-h)^2}{H^2}\cdot h$

$\therefore V = \dfrac{\pi R^2}{3H^2}(H-h)^2 h$

$\therefore \dfrac{dV}{dh} = \dfrac{\pi R^2}{3H^2}[(H-h)^2 - 2h(H-h)]$

$= \dfrac{\pi R^2}{3H^2}(H-h)(H-h-2h)$

$\therefore \frac{dV}{dh} = 0$ if $h = \frac{H}{3}$

and $h = \frac{H}{3}$ is a point of maximum $\Rightarrow \frac{H}{h} = 3$

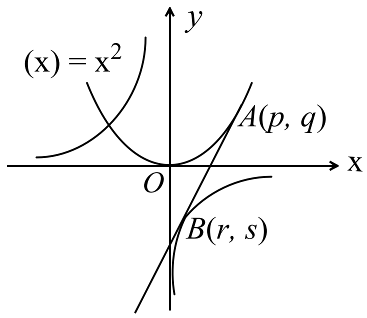
330 **(5)**

$y = x^2$ and $y = -\frac{8}{x}$; $q = p^2$ and $s = -\frac{8}{r}$ (1)

Equating $\frac{dy}{dx}$ at A and B, we get

$2p = \dfrac{8}{r^2}$

$\Rightarrow pr^2 = 4$ (1)



Now $m_{AB} = \frac{q-s}{p-r} \Rightarrow 2p = \frac{p^2+\frac{8}{r}}{p-r}$

$\Rightarrow p^2 = 2pr + \dfrac{8}{r} \Rightarrow p^2 = \dfrac{16}{r}$

$\Rightarrow \dfrac{16}{r^4} = \dfrac{16}{r} = \Rightarrow r = 1\ (r \neq 0) \Rightarrow p = 4$

$\therefore r = 1, p = 1$

Hence $p + r = 5$

331 **(8)**

Let $f''(x) = 6a(x-1)(a > 0)$

$\Rightarrow f'(x) = 6a\left(\dfrac{x^2}{2} - x\right) + b = 3a(x^2 - 2x) + b$

Given $f'(-1) = 0$

$\Rightarrow 9a + b = 0 \Rightarrow b = -9a$

$\Rightarrow f'(x) = 3a(x^2 - 2x - 3) = 0$

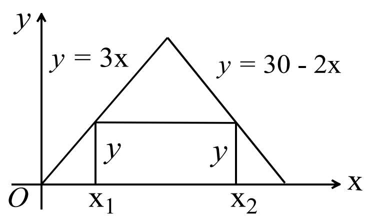
$\Rightarrow x = -1$ and 3

So, $y = f(-1)$ and $y = f(3)$ are two horizontal tangents

$\Rightarrow$ Distance between theses tangents =

$|f(3) - f(-1)| = |-22 - 10|$

332 **(5)**



$A = (x_2 - x_1)y$

$y = 3x_1$ and $y = 30 - 2x_2$

$A(y) = \left(\dfrac{30-y}{2} - \dfrac{y}{3}\right)y$

$6A(y) = (90 - 3y - 2y)y = 90y - 5y^2$

$6A'(y) = 90 - 10y = 0$

$\Rightarrow y = 9; A''(y) = -10 < 0$

$x_1 = 3; x_2 = \frac{21}{2}$

$\Rightarrow A_{\max} = \left(\frac{21}{2} - 3\right) 9 = \frac{15.9}{2} = \frac{135}{2}$

333 **(9)**

$y = x^3 + x + 16$

$\left(\frac{dy}{dx}\right)_{x_1, y_1} = 3x_1^2 + 1$

$\therefore 3x_1^2 + 1 = \frac{y_1}{x_1}$

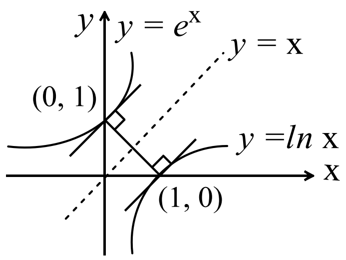
$\Rightarrow 3x_1^3 + x_1 = x_1^3 + x_1 + 16$

$\Rightarrow 2x_1^3 = 16$

$\Rightarrow x_1 = 2 \Rightarrow y_1 = 26$

$\therefore m = 13$

334 **(8)**



Since the graphs of $y = e^x$ and $y = \log_e x$ are symmetrical about the line $y = x$, minimum distance is the distance along the common normal to both the curves, i.e., $y = x$ must be parallel to the tangent as both the curves are inverse of each other

$\left.\frac{dy}{dx}\right|_{x_1} = e^{x_1} = 1$

$\Rightarrow x_1 = 0$ and $y_1 = 1$

$\Rightarrow A \equiv (0, 1)$ and $B \equiv (1, 0)$

$\Rightarrow AB = \sqrt{2}$

335 **(4)**

$x^2 - 2x - 3 > 0$

$\Rightarrow (x - 3)(x + 1) > 0$

$\Rightarrow x < -1$ or $x > 3$ (1)

Now, $f(x) = \log_{1/2}(x^2 - 2x - 3)$

$= \frac{\log_e(x^2 - 2x - 3)}{\log_e(1/2)}$

$f'(x) = \frac{2x - 2}{(\log_e(1/2))(x^2 - 2x - 3)}$

For $f(x)$ to be decreasing $f'(x) < 0$

$\Rightarrow \frac{x - 1}{(\log_e(1/2))(x - 3)(x + 1)} < 0$

$\Rightarrow x > 1$ (2)

From (1) and (2); $x > 3$

336 **(7)**

$x = t^2; y = t^3$

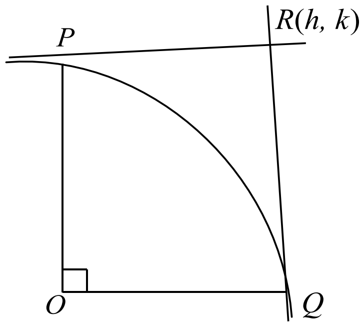
$\frac{dx}{dt} = 2t; \frac{dy}{dt} = 3t^2$

$\frac{dy}{dx} = \frac{3t}{2}$

$y - t^3 = \frac{3t}{2}(x - t^2)$

$2k - 2t^3 = 3th - 3t^3$

$\therefore t^3 - 3th + 2k = 0$ (1)



$t_1 t_2 t_3 = -2k$ (put $t_1 t_2 = -1$); hence $t_3 = 2k$

Product of roots

Now t_3 must satisfy equation (1)

$\Rightarrow (2k)^3 - 3(2k)h + 2k = 0$

$\Rightarrow 4y^2 - 3x + 1 = 0$ or $4y^2 = 3x - 1$

$\Rightarrow a + b = 7$

337 **(5)**

$y = e^{a+bx^2}$, passes through $(1, 1)$

$\Rightarrow 1 = e^{a+b}$

$\Rightarrow a + b = 0$

also $\left(\frac{dy}{dx}\right)_{(1,1)} = -2$

$\Rightarrow e^{a+bx^2} \cdot 2bx = -2$

$\Rightarrow e^{a+b} \cdot 2b(1) = -2$

$\Rightarrow b = -1$ and $a = 1$

$\Rightarrow 2a - 3b = 5$

338 **(1)**

$f(x)$

$= \begin{cases} |x^3 + x^2 + 3x + \sin x|\left(3 + \sin\left(\frac{1}{x}\right)\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$

Let $g(x) = x^3 + x^2 + 3x + \sin x$

$\therefore f'(x) = 3x^2 + 2x + 3 + \cos x$

$= 3\left(x^2 + \frac{2x}{3} + 1\right) + \cos x$

$= 3\left\{\left(x + \frac{1}{3}\right)^2 + \frac{8}{9}\right\} + \cos x > 0$

and $2 < 3 + \sin\left(\frac{1}{x}\right) < 4$

Hence, minimum value of $f(x)$ is 0 at $x = 0$

Hence, number of points = 1

339 **(4)**

Let $x = r\cos\theta, y = r\sin\theta$

$\Rightarrow r^2(1 + \cos\theta \sin\theta) = 1$

$\Rightarrow r^2 = \frac{2}{2 + \sin 2\theta}$

$\Rightarrow r_{\max}^2 = \frac{2}{1}$